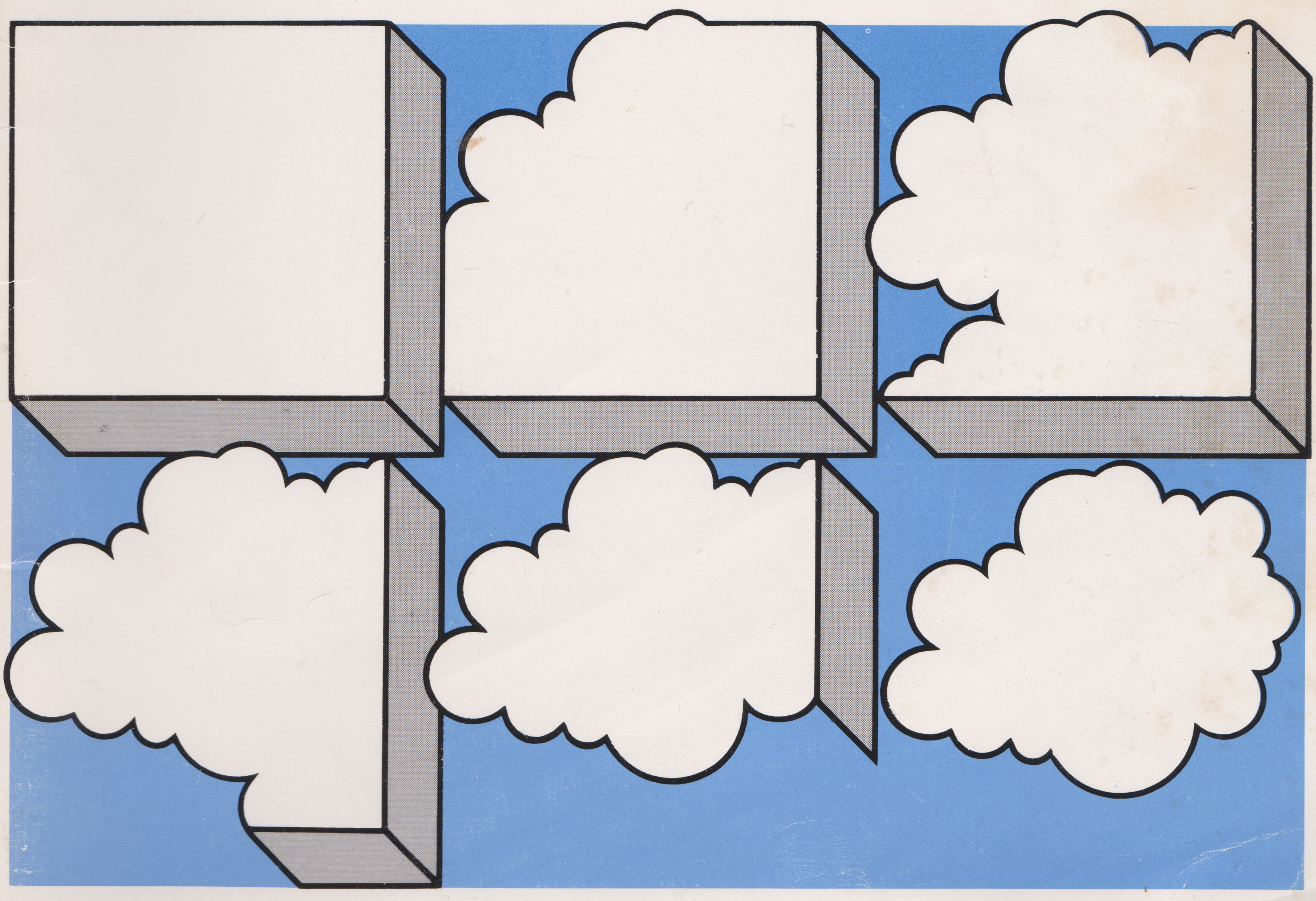
M101 LDM

THE OPEN UNIVERSITY
Mathematics Foundation Course

LEARNING AND DOING MATHEMATICS



THE OPEN UNIVERSITY

Mathematics: A Foundation Course

LEARNING AND DOING MATHEMATICS

Prepared for the Course Team by John Mason

Contents

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1.2

INTRODUCTION

Have you ever found yourself staring at a page of text with no idea what it is about?

Have you ever found yourself staring at a blank page of paper unable to start on a question?

The aim of this unit is to provide guidance on coping with these commonplace experiences, to assist you to learn to *do* mathematics rather than be confronted by it. There is no *new* mathematical content to be learned. The unit draws attention to specific activities or processes which lie at the heart of mathematical thinking. They are essential to learning mathematics from a text, and also to tackling mathematical questions. By the end of this unit, I hope that you will find that your awareness of the nature of mathematics and how to participate in it will be expanded.

The text contains numerous mathematical questions, and experience has shown that unless you make a serious attempt to think about them for yourself, the advice and comments will be of little help.

The unit focusses on the two processes of specializing and generalizing, which are fundamental to mathematical thinking, and indeed to thinking of any kind. Specializing means looking at particular cases of a general statement, and generalizing means abstracting features common to several particular examples, but the meanings of both words will expand as the unit progresses. There is nothing mysterious about these processes; indeed, we have all been going through them since birth. Learning to speak, to recognize people and shapes, and to read, are but three examples.

If these processes are as common as claimed, what is to be gained by drawing attention to them? One answer is that mathematical thinking makes constant use of them, and the more aware we are of what is required, the easier it is to engage in learning effectively. Although they are entirely natural acts, they are a little bit more subtle than they may first appear. For example, despite their naturalness, we all fail to use them when they would help most, and this failure accounts for a lot of the time that we spend being stuck. Although specializing and generalizing are hard to keep separated, the first section introduces some salient aspects of specializing, the second section looks at generalizing, and the third shows them in action together. Subsequent sections use them both in various guises, implicitly and explicitly.

Interlaced with the five main sections are five short interludes, which explore different ways of talking about studying and doing mathematics. They overlap considerably, because I believe that richness and clarity come from a multiplicity of perceptions. The interludes are very much one person's view, though they are based on considerable experience of difficulties that students encounter.

I recommend that you work through at least the first three sections during the study week allocated, and that you work through Section 4 before Summer School. I hope that you will find this unit useful, not just on first reading, but to return to at various times during this course and throughout your studies, whenever you feel the need to stand back and reflect on what it is you are really doing when learning mathematics.

SECTION 1 SPECIALIZING

The process of specializing is fundamental to mathematical thinking, indeed to thinking of any kind. Basically, specializing means looking at special or particular cases of some general statement, but since it is useful to look at cases which involve only familiar and confidently manipulable objects, such as

diagrams, numbers and so on, specializing quickly becomes associated with concrete, confidence-inspiring examples.

The purpose of specializing changes over time. Initially, it is done to try to understand what a statement or question is saying, and later it provides fodder for the reverse process of generalizing.

There is only one sensible thing to do when faced with a claim like the following.

> The sum of the cubes of the first N positive integers
> is the square of the sum of those integers.

You must try some specific examples, to see what is actually being said.

TRY SOME NOW

Here is what I found, but this is of no use unless you have already tried the problem yourself!

$N = 1$ The sum of the cube of the first positive integer is 1^3, which is 1.
The square of the sum of the first positive integer is 1^2, which is 1.

Not very inspiring or informative so far!
Now try it for $N = 2$ and $N = 3$.

$N = 2$ $(1^3 + 2^3) = 9$.
$(1 + 2)^2 = 9$ seems a bit more interesting.

$N = 3$ $(1^3 + 2^3 + 3^3) = 36$.
$(1 + 2 + 3)^2 = 36$, so at least the content of the claim is beginning to be clearer.

The initial reason for trying these three examples is *not* so much to see if the statement is correct, but to see *what* it is saying; to get it off the page and inside me, so that I can hope to make sense of it. The question of whether it is correct will come later, but in the meantime it must be stressed that just because the claim is correct in those three particular instances, it does *not* mean that it is necessarily always going to be correct for *any* positive integer N.

The entirely natural act of trying specific examples is called SPECIALIZING. It may seem so obvious in this case as to scarcely warrant a name, but there are two good reasons for drawing attention to it. Not only is it rather more subtle in some of its manifestations than may appear at first sight, but also one of the main causes of being stuck while doing a question or while reading a text is *failure to specialize appropriately*.

For the moment, attention should be on the nature of specializing, which in the case above involved using numbers to make sense of a verbal statement. However, specializing does not always involve numbers. Consider, for example, the formula

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

There are literally hundreds of formulas like this one in trigonometry, and they can be a bit daunting, to say the least. Yet no mathematician would dream of memorizing them all. For example, specializing by putting $B = A$ gives the formula

$$\cos(A + A) = \cos(A)\cos(A) - \sin(A)\sin(A).$$

Specializing

This formula is a special case, or specialization of the first formula, because the scope of the variables A and B has been constrained by the extra condition that A and B are to have the same value. Since this formula can be so easily obtained from the first, it makes no sense to memorize both of them. Similar specializations such as putting $B = 2A$ or $B = 3A$ will yield other formulas, all based on the original (as was done in Block I *Unit 3*). The more adventurous substitution, $A = (X + Y)/2$ and $B = (X - Y)/2$, which is a form of specializing, yields still fancier formulas. It is much easier to remember the main formula, and then specialize to get any of the others when they are needed, than to try to remember them all and run the risk of getting them mixed up.

As a further indication that specializing is not confined to numbers alone, try this next question, and see if your natural reaction is the same as mine.

> **Tethered Goat**
>
> A goat has been tethered with a 6 m rope to the outside corner of a 4 m by 5 m shed. What area of ground can the goat cover?

—————— TRY IT NOW ——————

My immediate reaction was that I needed a diagram. When I drew it, I found that the question seemed much more concrete. I could see more clearly what was involved, and I found myself extending the drawing to show the various regions that the goat could reach. This in turn showed me what calculations were needed. Drawing a diagram is a form of specializing, in that it is making the question more concrete, more specific, and more manageable.

Specializing

The *Tethered Goat* example shows that I am using the word "specializing" to refer to much more than simply trying numbers. It carries a sense of reduction to something simpler, something easier to work with, and so I include under specializing the movement from general to specific, even if the specific is a diagram, or some other representation which makes the question or statement easier to think about. The next example illustrates several sorts of specializing.

In Block I *Unit 2*, the claim is made that

> for any positive integer N, and $A \in \{0, 1, 2, \ldots, N-1\}$, if A and N have no common factor other than 1, then A has a multiplicative inverse modulo N.

The grammatical structure of the statement is a little tricky, so the most sensible thing to do is to specialize, to see what is going on.

—————— SPECIALIZE NOW ——————

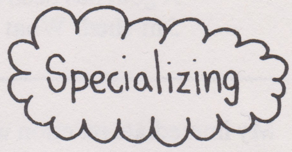
I chose $N = 12$ to start, because it has several divisors, but is not too large. That means A must be in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\ 10, 11\}$, and yet have no common factor with 12 other than 1. Consequently, A could be 1, 5,7, or 11. In the process of finding A, I have been reminded about common factors.

Now I need to know about multiplicative inverses. Specialize to ordinary numbers, where I feel happier—the multiplicative inverse of 2 is $\frac{1}{2}$, because their product is one. So what does a multiplicative inverse for A modulo N mean? It must mean some number B, such that $BA = 1$ modulo N. For my specialization, $N = 12$ with $A = 7$, B must be… . (Try each possibility in turn, or look for a number of the form $12N + 1$ which is divisible by 7.)

The extensive work with pins and thread in Block I *Unit 2* provided special examples of this sort of idea in a diagrammatic/physical context. It was included in order to assist you to get a real feeling for what is going on in modular arithmetic.

Specializing

One point about specializing which is often overlooked, is that it means resorting to particular examples which are confidence-inspiring for *you*. For example, other people's diagrams are never as informative as your own. Thus, what is specializing to one person may be abstract to another—it is all relative to your own experience and confidence. The goat example is pretty straightforward once a diagram is drawn, but it illustrates the general principle that it is often essential to find some way to make a question or a text more concrete. What does it mean to make something concrete? It means to specialize using objects with which you are familiar and which you can confidently manipulate. Sometimes it helps to use physical objects, sometimes numbers are what is needed, and sometimes it may be a matter of reducing complexity by using two-dimensional objects instead of three-dimensional ones, or one-dimensional instead of two. As your mathematical sophistication grows, the "objects" which give you confidence, and to which you resort for specializing, will become increasingly abstract (as viewed by a novice).



An example of levels of sophistication is given by the following statement:

$$^nC_r + {}^nC_{(r-1)} = {}^{(n+1)}C_r.$$

What on earth does it mean? I personally find ^nC_r an off-putting sort of notation, so I immediately specialize by using factorials, which I can confidently manipulate, i.e.

$$^nC_r = \frac{n!}{r!(n-r)!}.$$

I refer to this as specializing because it turns what is for me a succinct, abstract and not very confidence-inspiring symbol, into something easier to handle. I can now specialize the general formula by interpreting each term of the original statement in the same way:

$$\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!}.$$

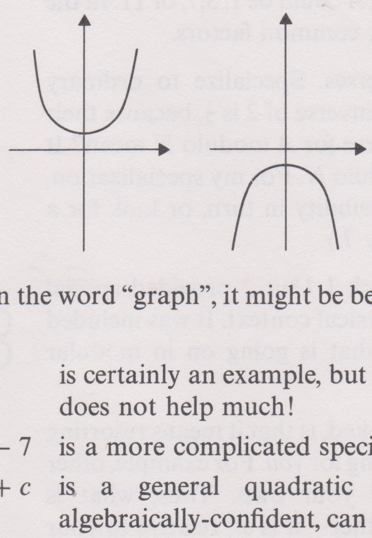
It may be that this is no more illuminating for you than the original, but remember that specializing is a relative term. If this is not concrete enough, make it more so! Try writing out what the factorials mean, try specializing further by putting in a specific number for r, try specializing even further by giving n a value and by confirming that the original expression is indeed satisfied by this choice of n and r. Effective specializing means using mathematical entities that *you* can confidently manipulate.

As a final example of this use of specializing to interpret a statement and make sense of it, consider the following quotation from M101 Block II *Unit 1*.

> Any quadratic graph can be obtained from the graph of $y = x^2$ by means of scalings and translations.

What does it mean? There are some technical terms which may or may not inspire confidence, but the only way to find out is to specialize, to try some specific examples. To do that, it is necessary to sort out the meanings of words, again by means of examples.

> What is a quadratic graph?



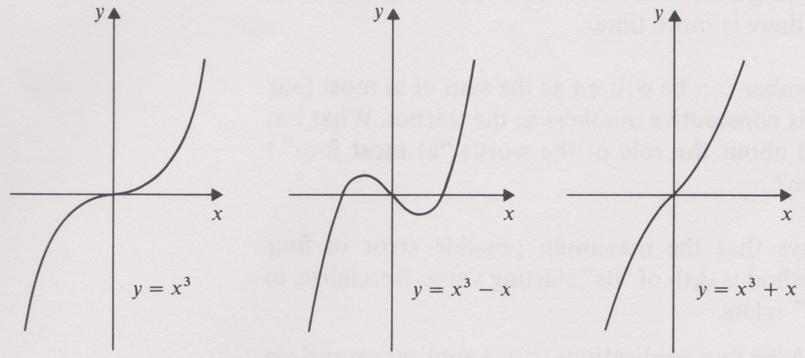
Before concentrating on the word "graph", it might be best to consider the word "quadratic".

$y = x^2$	is certainly an example, but it is the one given, so it does not help much!
$y = 2x^2 + 3x - 7$	is a more complicated specific example.
$y = ax^2 + bx + c$	is a general quadratic which, for someone algebraically-confident, can serve to focus attention on the meaning of "quadratic". It can be manipulated exactly as any specific "quadratic" can be manipulated.

The act of writing down some examples has reminded me of my Block II experience with such graphs, so other related ideas are beginning to come to mind—shapes of graphs, graphs being related to each other by geometrical transformations, etc. In particular, the ideas of scaling and translation are re-awakened, but if they were not, I would look back and find some examples, some way to write them down in a language with which I feel comfortable.

This example reveals another aspect of specializing, which runs throughout mathematics. The Block II discussion of quadratics focussed on the graph of

$y = x^2$, not just because it is a simple example of a quadratic, but because in a sense it can stand for, or represent, all quadratics. More precisely, it was shown that *every* quadratic looks like or can be obtained from $y = x^2$ by suitable scaling and translating. So $y = x^2$ is a very sensible particular example of a quadratic, well worth getting to know intimately. It can then be used as a confidently-manipulable example whenever needed. When the idea of a quadratic is extended to a cubic, it turns out that there is no one cubic that represents all cubics, rather there are three of them. Any cubic looks like, or can be obtained from one of these three, so there are three worth mastering and using as specializations of general statements about cubics.



So far, specializing has been recommended as an activity for getting in touch with what a question or statement says. There is a little more to it than that. In order to see what a statement really means, it is necessary to link the particular cases to the general statement that spawned them. In the first example, of the sum of the cubes, the statement does not merely claim that

$$1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2,$$

but that, more *generally*, the sum of the cubes of the first N positive integers is always equal to the square of the sum of those integers. The special case gives us a sense of what is meant. Together with other special cases, it suggests a pattern which captures the essence of the original. The understanding comes from the specializing. The process of seeing and describing the general pattern is generalizing.

Summary

It is often tempting to rush into using symbols, only to get bogged down in a morass of vaguely-grasped generality, endlessly manipulating the symbols to no effect. When you recognize this state of being STUCK, stop, relax, and accept that you are stuck, and go back and specialize! Knowing that you *can* specialize should begin to provide confidence so that you can treat being stuck as the honourable state that it is, the state from which so much can be learned as long as you do not panic.

Specializing is something which anyone can do in the face of almost any question or text. It is an entirely natural response to meeting an abstract statement, yet all too often it is overlooked. Whenever you find yourself STUCK, ask yourself if you have done enough appropriate specializing.

What does *appropriate* specializing mean? It means specializing drastically enough so that what you obtain makes use of confidence-inspiring, manipulable objects. Sometimes it helps to use physical objects, and sometimes numbers or diagrams are what is needed. As mathematical sophistication grows, so will the complexity of what is considered confidently manipulable:

numbers,

algebraic expressions,

functions,

sets of functions,

sets of sets of functions,… .

Mathematics is abstract only to the extent that you find yourself dealing with entities that are not confidence-inspiring—so specialize!

The purpose of specializing is to gain clarity as to the meaning of a question or statement, and then to provide examples which have some general properties in common, so that you can begin to see and appreciate those common properties—the process of generalizing.

Exercises

Choose two or three of the following questions which seem appealing, and come back to the others later, when there is more time.

1.1 It is claimed that every number can be written as the sum of at most four squares. Demonstrate this for six consecutive numbers in the thirties. What has your specializing demonstrated about the role of the words "at most four"? Could four be replaced by three?

1.2 In Block I *Unit 1*, it says that the maximum possible error of four applications of the bisection method is $\frac{1}{16}$th of "its" starting value. Specialize, in order to work out to what "its" refers.

STUCK? Specialize by simplifying four applications to one application and see what is being said. Then build up to two, three, four. The question is only asking to what "its" refers!

1.3 What is the square root of 12345678987654321?

STUCK? Specialize to shorter numbers of the same pattern!

1.4 Tartaglia and De Bouvelles

It is said that in 1556, Nicolo Tartaglia conjectured that the numbers

$$1 + 2 + 4, \quad 1 + 2 + 4 + 8, \quad 1 + 2 + 4 + 8 + 16, \ldots$$

are alternately prime and composite (i.e. not prime).

It is also said that in 1509, Charles De Bouvelles conjectured that one or both of $6N + 1$ and $6N - 1$ (N a positive integer) are prime for all N.

Specialize systematically in order to see what support there is for these conjectures.

1.5 Centre of Gravity

It is claimed that any line through the centre of gravity of a planar two-dimensional object will bisect the object into two regions of equal area. Comment.

STUCK? Start with familiar two dimensional objects. What is it about them that seems to make the claim work? What can be modified to stop it working? Be extreme in your specializing.

1.6 Conversion

I heard someone on the radio say that to convert Centigrade to Fahrenheit in this country, you need only multiply by 2 and add 30, and you'll be correct to within one degree. Comment.

STUCK? Try some values, and compare them with the true conversion formula. What is the force of "this country"?

1.7 Divides

If a number divides the product of some numbers, does it necessarily divide one of those numbers? Sort out what is being said, by specializing, and comment on the validity of the statement.

1.8 Unequal

If $x < 3$, then $x^2 < 9$. Specialize, with a view to checking the validity of this statement.

STUCK? Specialize by using numbers. Specialize by using a diagram.

1.9 Annulus

How many equilateral triangles are needed to make a true annulus of triangles if the triangles must be glued together along complete edges?

INTERLUDE A ON CONJECTURING

Mathematics is commonly thought of as a subject concerned solely with obtaining right/wrong answers. It must be stressed that this is a

MISCONCEPTION!

Certainly, there is an aspect of mathematical work concerned with obtaining particular answers to particular questions, but this is really a very small part of learning mathematics—the tip of the iceberg. All that is visible to most observers is the outer manifestation of answers to questions. The inner activity is like the rest of the iceberg—submerged from view, and by far the largest and most important part.

Take, for example, the notion of a function. It is a technical term in mathematics, and there is a formal definition for it, but when the word "function" is encountered, all sorts of associations are available to support the formality of the definition. Notions such as

a process typified by the arrow $x \longmapsto$,
a graph,
a formula $f(x)$,
some examples, ...

all accompany the idea of a function. It is not simply an abstract concept, because it has a component that can be felt in the body as a movement from domain to image, it has a visual component in the form of graphs, and a symbolic component in the form of formulas and the idea of x as a slot to be filled.

There is no right "sense" of what a function is, though there can be inappropriate or limited senses. Thus for several generations of mathematicians, the sense of "function" meant a polynomial such as

$$x \longmapsto x^3 + 3x^2 - 7.$$

It took a long time to accept that something like

$$x \longmapsto \begin{cases} x^2, & \text{if } x > 0 \\ x^3, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is also a function. It certainly satisfies our definition of a function, since each real number in the domain is mapped to a unique real number in the codomain, yet it can come as a bit of a shock that functions can be pieced together in this way—it opens the door to all sorts of other possibilities! Most of the ideas in mathematics involve multiple ways of looking at something, and there is no one right way.

The fact that mathematics is full of multiple ways of looking at things carries over into the way mathematicians work. Most of the time, mathematicians are very tentative people. They are much more inclined to make a conjecture, that is, to say something which they have every intention of modifying, than they are to make definitive statements that are supposed to be true and correct. This humble attitude rarely comes across in texts, yet many dead-ends may have been encountered and abandoned, and many attempts to say clearly what is meant will

have been scrapped before the text is ready. The process of sorting out ideas, whether initiated by a text or by working on a question, involves making conjectures, and being happy to modify them or even throw them out all together if necessary.

Mathematical thinking is best supported by adopting a conjecturing attitude. Never be afraid to offer a tentative conjecture about something, but equally, *do not believe* your conjecture. As soon as I have made a conjecture, I try to write it down or say it out loud. Getting it out of my head and onto the page in front of me helps me to clarify my thoughts. If I try to keep it in my head, it remains woolly and half-formed, and it will certainly clog up my thinking. The struggle to get it down on paper refines and clarifies my conjecture, and clears space so that I can look at it coolly and objectively, checking it on examples to see if it seems reasonable, and trying to find out why it might be inadequate. When it turns out not to be quite right, then I am happy to modify it. That is how mathematicians operate all the time.

The same conjecturing approach applies when working on assignment questions—your rough notes should contain numerous conjectures and modifications. When you have done all you can, it may be that you are uncertain about some parts. The best strategy is to write down your tentative conjecture, clearly marking it as such, and/or to ask your tutor, in writing, for assistance. Tutors marking scripts are always looking for reasons, even excuses for giving marks. They are never trying to find ways of removing them. Put another way, assignments and examinations are opportunities for you to show what you *can* do, not hide what you cannot. If you indicate uncertainties on an assignment, the tutor can be much more helpful to you than if you succeed in covering up your ignorance and fooling the tutor. Thus, in the middle of a computation, a student came across

$$x^2 < 9.$$

Uncertain what to do, he wrote

$$\text{therefore } x < 3.$$

On paper it looks as though he believes it, whereas in fact it had the status of a weak conjecture. Given that there is not time to think it through by specializing, indicating clearly its status as a conjecture is much more likely to impress the examiner. Furthermore, it is important to gain confidence in your own assessment of what you know. When you feel unsure, it is probably because something is not quite right. The mathematical approach is to register that uncertainty, by labelling the statement as a conjecture. Then if there is time, you can come back and check it out. The chances of fooling the examiner are much smaller than the chances of impressing her!

The purpose of this interlude is to stress that an attitude or an atmosphere of conjecturing frees you from the dreadful fear of being wrong. In fact, it is often better to be wrong so that you can modify your statement and hence your understanding, than it is to be right but perhaps for wrong reasons. In other words, we should bless our mistakes as golden opportunities. Being stuck is an honourable state from which much can be learned. Being right lessens the opportunity to modify and learn. There is always a temptation, when working in a group, to sit back and let others do all the talking in case you make a fool of yourself. In a group which is working mathematically, quite the reverse should be the case. Everyone should be encouraged to express what they have understood, or what they think might be true, so that others can question, and invite or suggest modifications.

Advice

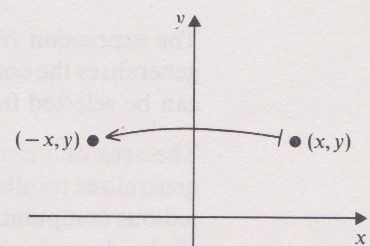
Always mark conjectures *as* conjectures in rough working, assignments and even examinations. That way you will be able to see at a glance what needs further work or sorting out, if time permits, and tutors can be of specific direct assistance. If a group of you are meeting to talk, let it be the group task to encourage those who are *unsure* to be the ones to speak first, and more often than the ones who are sure. Help them and yourself by creating a supportive atmosphere, that is, one in which every utterance is treated as a *modifiable conjecture*!

SECTION 2 GENERALIZING

Generalizing is the reverse of specializing, and the two processes go very much hand in hand. Generalizing has to do with noticing patterns and properties common to several situations. A generalization is an expression or statement which can be specialized. It need not actually be true, but it must be able to be specialized to produce the particular cases which it generalizes, amongst others. For example, the sequence of symbols

$$(x, y) \longmapsto (-x, y)$$

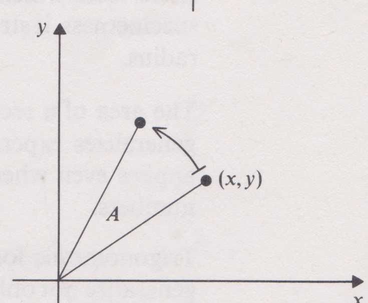
is a way of writing down the effect of a reflection in the y-axis. It captures the essence, the property that is common to all particular instances of a point in the plane being reflected in the y-axis.



Similarly,

$$(x, y) \longmapsto (x\cos A - y\sin A,\ x\sin A + y\cos A)$$

is a general description of the effect of rotating points in the plane, through an angle A anti-clockwise about the origin. It summarizes the effect on any particular point being rotated through any particular angle.



The statement

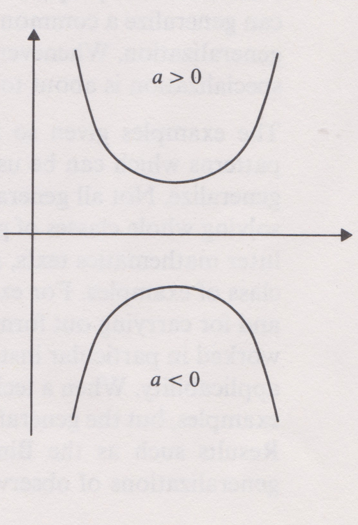
at $x = -b/a$, $ax^2 + bx + c$ achieves its maximum value if a is negative and its minimum value if a is positive

is a general statement about the maximum and minimum value of any quadratic expression. It states something which is true about each particular instance, of a quadratic, for example

$2x^2 - 3x + 4$ has a minimum at $x = \dfrac{3}{2}$.

$-3x^2 + 4x - 1$ has a maximum at $x = \dfrac{4}{3}$.

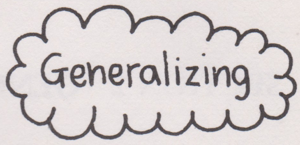
$\pi x^2 - x + \sqrt{2\pi}$ has a minimum at $x = \dfrac{1}{\pi}$.



In each case, the sequence of symbols should evoke the awareness of specializations, and also some sort of image.

Such general statements make sense, and are of use, only if they act as a catalyst to crystallize experience, fusing together what previously was a lot of disparate examples, both geometric and algebraic. Notice also that a diagram is a particular case, but it illustrates the generality as long as certain features are stressed, such as shape, and certain other features are ignored, such as position and axis-scales.

Often when we feel that we have understood something, it is because we have become aware of connections with other things with which we are already confident. Rather than accumulating a library of facts, it is much easier and more appealing to try to find a general principle which accounts for large numbers of specific instances. This is the role of the most common form of generalizing—the general formula. For example:



The quadratic formula $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$
gives the roots of the quadratic equation $ax^2 + bx + c = 0$. It is a general statement which can be obtained by generalizing the process of completing the square in any particular instance.

The expression $2N + 1$
generalizes the computation of the number of ways that a pair of objects can be selected from a particular number (N) of objects.

The expression $N(N - 1)/2$
generalizes the computation of the number of ways that a pair of objects can be selected from a particular number (N) of objects.

The area of a circle $= \pi \times r^2$
generalizes results obtained for particular circles by otherwise long and tedious computations involving approximations inside and outside the circle, ideas which led ultimately to the idea of an integral. Because of its succinctness, it stresses the fact that the area depends on the square of the radius.

The area of a rectangle = length × width
generalizes experience with counting squares that tile a rectangle. It applies even when the length and width are not integers but any real numbers.

Trigonometric formulas like the one for $\cos(A + B)$
generalize not only specific calculations that can be done with particular angles, but also relationships that hold between angles such as A and $2A$.

..., and so on.

Notice that the purpose of these examples is to provide enough data so that you can generalize a common quality: the quality that makes each a special case of a generalization. Whenever the words "for example" appear, it is a sure sign that a specialization is about to be given, which is supposed to help you to generalize.

The examples given so far suggest that formulas are expressions of general patterns which can be used to recover any of the specific instances which they generalize. Not all generalizations emerge as formulas, however. Techniques for solving whole classes of problems, and the dreaded theorems with proofs which litter mathematics texts, are also statements of general properties common to a class of examples. For example, techniques for solving simultaneous equations, and for carrying out formula iterations, are generalizations of techniques which worked in particular instances and which were then recognized as having wider applicability. When a technique is being learned, it is seen being applied to a few examples, but the general method should be applicable to many other situations. Results such as the Binomial Theorem and Pythagoras' Theorem are also generalizations of observations, synthesizing experience and intuition.

Of course, someone else's generalization is much less likely to have impact or significance than is your own, so whenever you encounter a general statement which seems opaque, or makes you feel uneasy, you should immediately

> specialize;
>
> try to see what the general statement is saying in the particular instances;
>
> try to reformulate the statement in your own words as your own generality.

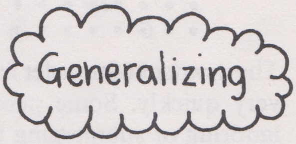
You can usually tell when this process has been successful, because a good deal of pleasure and lightening of spirits results from capturing a generality for yourself.

The same pleasure can be derived from questions as well as from book-work. Take, for example, the *Tethered Goat* question which was mentioned in the discusson of specializing. It seems like a very dull question, which is easily answered once a diagram is drawn. But what happens if we try setting it in a more general context? We can replace the numbers by letters, but this is only an elementary form of generalizing. More significant questions emerge when we try varying different aspects of the question: imagining ourselves in a field and looking for complications that might arise, and imagining ourselves in an ideal-mathematical sort of field and looking for alterations to the conditions. In short, generalizing means placing in a more general context.

TRY GENERALIZING IT NOW

The idea is to pose questions which involve loosening or changing some of the constraints which are either stated or implied by the original question.

The ideas that occurred to me are listed below. Not all of them are fruitful, or even sensible. There will be time enough to make judgements if and when I decide to try to resolve some of them!



> What about changing the size of the shed and the length of the rope?
>
> What about obstructions such as trees (mathematical ones would be points, or circles), or other sheds?
>
> What about a circular shed (I'm thinking of a silo)?
>
> What about polygonal sheds? (This was suggested to me by the circular shed, because I could use polygonal sheds to approximate a circular one.)
>
> What about three dimensions? I could look for volumes.
>
> What about fields with boundary fences as well?
>
> Instead of asking for the area the goat can reach, I could specify a particular fraction of the whole field of some shape, and ask what length of rope is required. For example, if the goat is tethered to the edge of a circular field, what length of rope is needed to permit it to graze exactly half the field?

This question is pursued in Block III *Unit 1*.

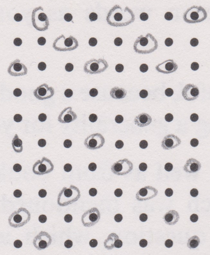
These questions will, I hope, give some idea of what is meant by generalizing a question, in the sense of placing it in a more general context.

Finding generalizations which unify previously disparate ideas is the principal source of pleasure in mathematics. It is a bit like finally finding a jigsaw piece which has eluded you for a long time. Putting it in place suddenly brings sense to an otherwise confused picture. Unlike specializing, which is almost always easy to do, generalizing is more of an art, because it involves noticing things that are common to numerous examples, and ignoring features which seem to be special to only some of them. The best way to learn about generalizing is to try it, so I recommend that you take every opportunity, both in this unit and in any course you study, to replace other people's generalizations by your own.

The following exercise is extremely simple and straightforward, but in many ways it typifies the experience of making a generalization.

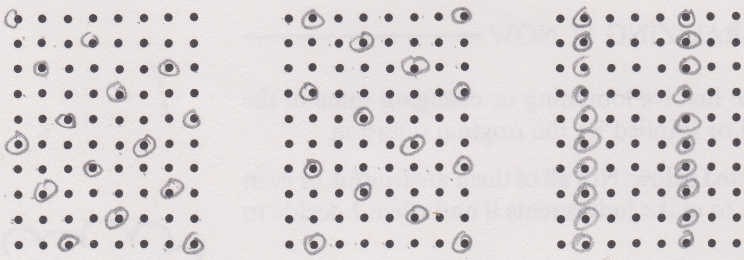
Dotty

Place a circle around every third dot, starting with the second and continuing from row to row, always reading from left to right.



DO IT NOW

I find that I begin by counting out carefully, and after a while something inside me takes over, the counting disappears, and I find myself quickly circling the dots in a diagonal pattern. If you did not notice this happening, try it again on one of the arrays below. Try circling every fifth dot starting from the third, try the same rule but reading right to left on alternate lines, and try your own rule.



The moment of transfer from tedious counting to perceived pattern often happens very quickly. Some people are cautious, and continue counting, steadfastly ignoring or submerging the general pattern, while others, at the other extreme, jump very rapidly to the diagonal pattern, but are prone to making errors because they have not checked their conjecture. The best method is probably somewhere in between, with flexibility.

The simplest generalizations are probably those connected with algebra. For example, the following.

Dinner

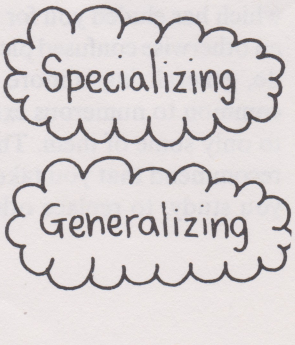
An ancient manuscript offers the following puzzle.
A lion can devour a sheep in 2 hours.
A wolf can devour a sheep in 3 hours.
A dog can devour a sheep in 5 hours.
How long would it take them to devour a single sheep, all eating at the same time? (Ignore the fact that the animals would squabble—medieval puzzles were rarely practical!)

GENERALIZE.

TRY IT NOW

(My resolution is on page 55.)

Calculations with the particular numbers given in the question point the way to a generalization in which the lion takes L hours, and so on. The authors of the original manuscript would have called it a rule or formula for solving such puzzles. By following the progress of 2, 3 and 5 through the arithmetic without actually *doing* the arithmetical calculations, the general statement can be written down by treating the numbers 2, 3 and 5 as place holders. By using symbols such as L, W and D instead of the numbers, it is much easier to see which places are being held for which things.



More animals can also be added, once the shape of the calculation is perceived. This is what generalization is about. Notice that the answer can be re-stated as

$$1/\text{TIME} = 1/\text{L} + 1/\text{W} + 1/\text{D} + \ldots,$$

Generalizing

which reminds me of the formula for adding parallel resistances in electrical circuits. Perhaps the sheep can be seen as some sort of resistance—an unexpected connection, which might be worth pursuing by looking for other questions which give rise to a similar expression. That is what is meant by seeing something in a more general context!

The idea of doing calculations with specific numbers in order to get a sense of what calculations to do in general is not confined to puzzles. For example, the following.

Ratios

If A, B, C, and D are positive real numbers, and if

$$A/B = C/D,$$

then

$$A/B = (A + C)/(B + D).$$

On first exposure, this sort of question seems quite surprising, and disbelief is best supported (or contradicted), by trying some specific numbers.

DO SO NOW

I found that choosing values for A, B, C and D gave me a clue as to how to proceed in general. Taking

Specializing

$$1/2 = 3/6 \quad \text{gave me} \quad (1 + 3)/(2 + 6),$$

and from somewhere deep inside me came a sense of the 2 being related to the 1 in the same way as the 6 to the 3, so that adding makes no difference.

I wanted some way to record that relationship, so I decided to let

Generalizing

$$T = A/B = C/D.$$

Thus $A = TB$ and $C = TD$, which is how I *felt* the relationship. Having the confidence to try to capture in symbols my sense of what was going on is an essential element at this point. Such confidence can be gained by getting into the habit of specializing so that you always know that there *is* something you can do when you get stuck.

Now I could check that the ratio has the value I wanted:

$$\begin{aligned}(A + C)/(B + D) &= (TB + TD)/(B + D)\\ &= T.\end{aligned}$$

This just says, in algebraic language, what I noticed in my specialization.

The only reason for offering this example is that it shows how using numbers can reveal a structure that is not so obvious, unless you already find algebra as easy as arithmetic.

Before dismissing this little question, it is worth testing the extent of your perception of its generality by taking a moment to write down a general statement which includes *ratios* as a special case. You might also like to consider what happens if you let the equality change to an inequality.

GENERALIZE RATIOS NOW

(My resolution is on page 55.)

Investigating the structure of numbers produces all sorts of surprises, and draws upon generalizing in subtle ways. For example:

1, 3, 5, 7,... are all the positive odd numbers;

I can represent any positive odd number by the expression $2N + 1$ (N a positive integer) which stands for a general odd number.

Here, $2N + 1$ generalizes the pattern of odd numbers. The "$+1$" captures the property of odd numbers being one more than an even number, and the "$2N$" captures the property of an even number as a number divisible by 2. Thus $2N + 1$ is a general odd number. By convention, the N signals the presence or availability of a positive integer (or zero), and in that sense $2N + 1$ describes all positive odd numbers. By convention, N can also mean all integers both positive and negative, and then $2N + 1$ represents *all* odd numbers. If N is read as standing for some *particular* but unspecified integer, then $2N + 1$ represents any particular but unspecified odd number. Expressions like $2N + 1$ are useful for representing general odd numbers when an argument is sought to show that the sum of any two odd numbers must be even. It is tempting to write $(2N + 1) + (2N + 1)$, but this is not sufficiently general, because it represents the sum of two copies of the same arbitrary but unspecified odd number. The argument requires an expression like $(2N + 1) + (2M + 1)$. The fact that the sum is even emerges because the expression can be rewritten as $2(N + M + 1)$.

There are many other ways of writing down a general description of odd numbers; for example $2N - 1, 2N + 3, 2N - 7$ are all equally general, and useful in some circumstances. Now try the following similar example yourself.

Remainders

Write down a general description of all integers which leave a remainder of 1 when divided by 5. Write down a description of all integers that have remainder 3 when divided by 7.

TRY IT NOW

STUCK? Have you specialized first?

Following the model provided by $2N + 1$ for odd numbers, $5N + 1$ and $7N + 3$ are reasonable candidates. The exercise should not be left there, however, because even the simplest of observations can lead to rich mathematical ideas. When I start thinking of odd and even numbers, I am reminded that the product of two even numbers is even, and the product of two odd numbers is odd. It occurs to me, that since $5N + 1$ is analogous to an odd number, I should try to set this observation in a more general context.

GENERALIZE NOW

What occurred to me is that since remainders on dividing by 5 is a variation on odd and even numbers, it might be interesting to ask the following questions.

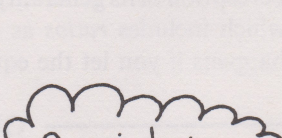
Which remainders modulo 5 have the property that any two numbers with that remainder, when multiplied, give a product with the same remainder?

Focus on the process of investigation and not on the content!

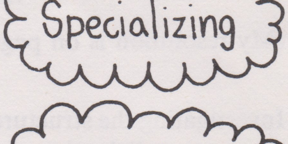
More generally, which remainders R modulo M have the property that any two numbers with remainder R, when multiplied, give a product with remainder R?

TRY IT NOW

The following is an account of my investigation of this question. I recommend that you try working on the problem yourself before reading my version.



Dealing with arithmetic modulo 5 first, the numbers with remainder R are described by $5N + R$. I want to find the product of two numbers of this form, and see what conditions are imposed on R by demanding that the product has the same remainder.



Two arbitrary numbers with remainder R are $5N + R$ and $5M + R$. (It would be tempting to take $5N + R$ and multiply it by $5N + R$, but this would represent the square of a number, not the product of *any* two numbers with remainder R.)

$$(5N + R)(5M + R) = 25NM + 5NR + 5MR + R^2.$$

I want this product to be in the class of numbers which have remainder R when divided by 5. The first three terms in the product are all divisible by 5, so the remainder on dividing the whole expression by 5 is R^2—no it's not, it's the remainder on dividing R^2 by 5. Thus I want

Note the quickly modified conjecture!

R^2 to have remainder R when divided by 5.

Any values of R which satisfy this, will answer my original question. I know now that R^2 is to have remainder R, and I want to find out what that tells me about R. It means that

$$R^2 = 5K + R \quad \text{for some integer } K,$$

and this means that

$$R^2 - R = 5K,$$

or, in other words,

$R^2 - R$ must be divisible by 5.

This means that

5 must divide $R(R-1)$,

and since 5 is a prime, 5 must divide R or $R-1$.

But what kind of a beast is this R? I know that it is a remainder upon division by 5, so it must be 0, 1, 2, 3 or 4. It follows that if 5 is to divide R or $R-1$, then R must be 0 or 1. In other words, the only two types of number are $5N$ and $5N+1$. Not trusting my algebra completely, it is worth checking $5N$ and $5N+1$ to make sure that they do work.

Check by specializing all possible values of R.

DO SO NOW

Looking back over what I have done, I discover that I have answered the particular question concerning division modulo 5, and the answer is that working modulo 5, the product of any two numbers with remainder R will also have remainder R only when R is 0 or 1. These two remainders correspond to the idea of evenness and oddness when working modulo 2.

I have dealt with remainders modulo 5, but what about other moduli? Read over the argument with an eye to whether 5 is essential, and if so, in what way. Could it be holding a slot for other numbers as well?

RE-READ IT NOW

I reckon that 5 could be replaced by 6, or 7, or any number at all (for instance M) in every line down to and including the line

5 must divide $R(R-1)$.

The next line uses the fact that 5 is prime. Thus 5 could be replaced by any prime number, and the argument will remain valid. I do not yet know what happens at the next line if 5 is replaced by a composite number, so I replace 5 by a general modulus M, and investigate.

I am now looking at

M must divide $R(R-1)$.

I can still conclude that $R=0$ and $R=1$ are solutions, but it may be that M factors in such a way that some of it divides R and some divides $R-1$. I need to specialize, to find examples of this. Since I now want M to divide $R(R-1)$, it seems reasonable to specialize R and look for examples of suitable Ms.

$R=3$, $\quad 3(3-1)=6 \quad$ so M could be 6 (it must be bigger than R by the definition of R).

$R=4$, $\quad 4(4-1)=12 \quad$ so M could be 6 or 12.

$R=5$, $\quad 5(5-1)=20 \quad$ so M could be 10 or 20.

$R=6$, $\quad 6(6-1)=30 \quad$ so M could be 10, 15 or 30.

I pause for a moment to wonder what I am doing. I wanted to find R given M, but now I find myself fixing R and looking at possible M. Certainly, there seem to be plenty of instances which I could go on investigating, but that is not the point of this section, which is about generalizing. I claimed earlier that putting even simple observations into more general contexts can lead to mathematical richness and pleasure, and that is borne out in this instance at least. I shall leave it by tidying up what I have found, so that if I return to it at some later time, I shall be able to pick up the threads.

I found that the product of two numbers with remainder R on division by M also has remainder R when R is 0 or 1. When M is 5, and more generally when M is prime, these are the *only* numbers having that property. When M is not prime, other numbers may also share this property. For example,

$M = 6$: $6N$, $6N + 1$, $6N + 3$, $6N + 4$;
$M = 10$: $10N$, $10N + 1$, $10N + 5$, $10N + 6$;
$M = 12$: $12N$, $12N + 1$, $12N + 4, \ldots$.

This example illustrates the richness of trying to see things in more general contexts, in opening them out and seeing what happens if restrictions are removed. It is noticing and articulating patterns which gives generalizing its beauty and its subtlety.

Summary

Generalizing is the twin process of specializing. It involves spotting a pattern common to a number of instances, and trying to express it. It often comes about from trying to see or set a result in a more general context, and this is why generalizing is so intimately connected with understanding. Generalizing is about making connections, and capturing them in a succinct statement from which particular instances can be retrieved by specializing.

Generalizing takes place whenever you stand back from what you are doing to try to distinguish the wood from the trees, trying to place it in a broader context. It is happening whenever you ask for sample questions to be worked by a tutor, in order to try to pick up a procedure for solving similar questions. It is an integral part of understanding, particularly in mathematics, and it is a source of great pleasure.

Whenever you find yourself stuck on a problem or a text, you should first specialize in order to penetrate the technical terms and the wording. Then you should look at the examples you have done and try to see what is common among them, guided by what the problem or text asks for or states. Even if the text is relatively clear, it is much more effective to try to state your own version of it for yourself. There is a great deal of pleasure available by doing this for yourself, and having done it once, it is remarkable how little effort is required to recall it again later.

Strictly speaking, a generalization of some examples is a statement that can be specialized to yield the examples again, but it is useful to carry with generalization the feeling of a "theme with variations". By changing conditions, both explicit ones and implicit ones, and by permitting constants to vary, a statement or question can be placed in a more general context, and interesting questions are likely to emerge. The person who has done the generalizing is the one who will most enjoy pursuing them!

One thing you have to be careful about when you get the generalizing bug, is that you are liable to start asking difficult questions very quickly! Generalizing for the sake of generalizing is like any other drug—you can soon lose touch with reality. The desire to generalize is no excuse for failing to specialize, to become thoroughly familiar with specific examples, since it is only through such intimacy that powerful generalizations will emerge.

Exercises

In each of the following, state a generalization. Make sure that it does, indeed, specialize back to the cases which spawned it. Do not spend time *now* trying to justify your conjectures.

2.1 $2 + 2 = 2 \times 2$ $\quad 2^2 \times \frac{2}{3} = 2 + \frac{2}{3}$

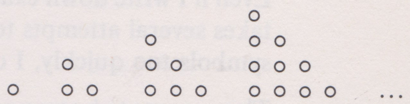
$3 + \frac{3}{2} = 3 \times \frac{3}{2}$ $\quad 3^2 \times \frac{3}{8} = 3 + \frac{3}{8}$

$4 + \frac{4}{3} = 4 \times \frac{4}{3}$ $\quad 4^2 \times \frac{4}{15} = 4 + \frac{4}{15}$

2.2 Given three positive numbers, the condition they must satisfy in order to be the lengths of the three sides of a triangle, is that each must be smaller than the sum of the other two.

Put this statement in a more general context, in several ways. Do not spend time trying to find out if they are correct. Leave them as questions.

2.3 How many dots are needed to make a triangular array with 37 rows like the ones shown with 1, 2, 3 and 4 rows?



STUCK? Try changing the question by putting two identical triangles together and predicting the number of dots needed for the new figure.

2.4
$$7^2 = 49$$
$$67^2 = 4489$$
$$667^2 = 444\,889$$

2.5
$$4^2 = 16$$
$$34^2 = 1156$$
$$334^2 = 111\,556$$

2.6
$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

2.7
$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

2.8 Since $\dfrac{4}{6} = \dfrac{10}{15}$, it follows that $\dfrac{4-6}{4+6} = \dfrac{10-15}{10+15}$.

Generalize!

INTERLUDE B ON CRYSTALLIZING

When an idea or technique is first encountered, it tends to be fuzzy, indistinct and imprecise. Gradually, as further experience is gained, it seems to take shape, until it reaches a reasonably stable form. Take, for example, the idea of a number. What does it mean to you?

As soon as I encounter the word number, I find a wealth of associations beginning to flow: some examples; some properties such as the fact that numbers can be added, etc.; and I am reminded that there are different sorts of numbers, such as whole, positive and negative, rational and irrational, and so on. The very fact that I have so many associations could be taken as an indication that, for me, the idea of number has crystallized around a sense of numbers like 2, $\sqrt{2}$ and π. There are also associations with diagrams such as "the real line". It is possible that I shall, at some later time, be forced to re-orient my number associations, just as I had to do when I first became aware of the idea of negative numbers, of fractions as numbers, and of irrational numbers. Arithmetic modulo 3, or more generally modulo any integer, also brought about a widening of my number horizions.

How do ideas become associated, and how can crystallization be induced? There are, of course, many facets, but one which deserves much more attention than it usually receives, has to do with activities which I can carry out to help me transform the ideas I meet in a course, into my own ideas. The basic principle is that in the early stages of encountering an idea, while it is still fuzzy, I *see* what it is about, long before I can talk coherently about it to someone else. Even after I can *talk* about it, I find it very hard to *write down* what I understand in a coherent fashion. This interlude is about the transitions from seeing, to saying, to recording.

When generalizing takes place, I find that there is an initial state, which may be very brief or very long, in which I feel that I can see a generalization, but I cannot say it or record it in any way. Subsequently, I realize what I am doing, and I begin to be able to say what the pattern is. If challenged to record the pattern in words or symbols, I find it very difficult, though sometimes diagrams make it easier. Even if I write down exactly what I say, it seems to come out very awkwardly. It takes several attempts to arrive at a sensible exposition, and if I try to move to symbols too quickly, I can get myself in a mess.

There seems to be tremendous resistance to making the transition from saying to recording. No matter how much stress a tutor places on getting students to write down notes to themselves about what they are doing when working on a text or a problem, students (and I include researchers) find it hard to do. After considerable observation of myself and others, I have come to the conclusion that there is a great difference between spoken speech and written records. Speech is halting and disjoint in the sense of mostly unfinished sentences, and it is constantly monitored and modified. Writing, on the other hand, demands complete sentences, and does not respond well to frequent qualifications and modifications, despite what one might expect. In other words, our reluctance to write things down is not simply due to laziness or some other obduracy, but reflects an inherent difficulty in switching modes.

It seems reasonable, therefore, to pay some attention to the transitions from seeing something, to saying what I see, to recording it. In particular, it often helps in the latter transition to use pictures and bits of sentences before trying to get complete thoughts, and especially before trying to put everything into symbols. It will be much easier to make records if I am articulate about what I have seen, so it is well worth spending time telling and re-telling someone (or something, like a goldfish or an imaginary friend walking with me to the bus stop!) how I see what is going on. The attempt to talk about some idea, and then to write it down, is an attempt to capture or crystallize it in some form so that it gets out of my brain. Then I can look at it afresh. So often, ideas can lie in a pre-articulate woolly phase, and then when I need them, I find that it all slips away. (How many times have you thought you understood something only to find a day or so later that it has all escaped?) Research mathematicians know that the very act of trying to tell someone else your problem helps to clarify things, and often the listener need say nothing.

Advice

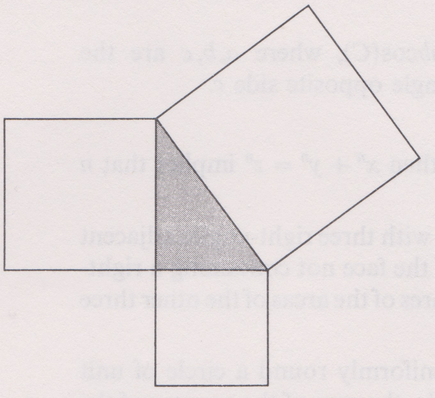
Whenever you encounter an idea in a text, try re-stating the idea in your own words. At first, you may find it hard to use words other than the ones in the text—if so, specialize to some examples, and tell yourself, out loud if at all possible, in what way the specializations are special cases of the more general idea. At the end of each section, try to tell yourself what you think the section has been about. When you have had several goes at saying it, try to capture it as succinctly as possible, using symbols, pictures and a few words. Refer back to these every so often, as you need them.

SECTION 3 SPECIALIZING AND GENERALIZING TOGETHER

Although the two processes were introduced separately, the discussion so far suggests that, in many ways, they are hard to separate out, in the sense that the reason for specializing is to permit and to promote generalizing, and that generalizations need to be checked in specific instances before looking for a convincing argument. To show how intimately they are bound together, this section begins by investigating the famous theorem of Pythagoras, which has been the starting point for many different strands of mathematics.

Pythagoras' Theorem

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.



Symbolically, this can be written out as the following.

If C is the hypotenuse of a right-angled triangle, and A and B are the other two sides, then

$$A^2 + B^2 = C^2.$$

There are several differences between the two versions. To begin with, the first version speaks of "the square on ..." meaning "the area of a square with side ...", but this is entirely lost in the second version. It is interesting to note that when symbols are employed, most of the statement is taken up by naming the symbols and indicating their role. Even so, this version fails to state that C is actually the length of the hypotenuse, and similarly for A and B. Since the equation is the bit which really interests us, it is a real benefit to have a succinct expression which we can easily modify and manipulate. Furthermore, most people remember the equation as the theorem and then reconstruct what the notation means, if necessary.

Even though the theorem bears the name of Pythagoras, who lived in the sixth century BC, there is considerable evidence that it was known, at least in some form, at least a thousand years earlier by the Babylonians, who stored tablets in their library containing at least fifteen specific numerical cases of the theorem, such as

$$3^2 + 4^2 = 5^2,$$
$$5^2 + 12^2 = 13^2,$$
$$8^2 + 15^2 = 17^2.$$

We do not know whether there were any pictures or diagrams to accompany the table. The fragment that remains contains integer solutions to the Pythagorean equation, and they are sufficiently extensive and organized to suggest that the Babylonians had some procedure for generating numerical examples, even if they did not have an explicit statement of the geometrical theorem. We also know of a Chinese geometrical instance of the theorem, of uncertain date, but at least as early as Pythagoras. Notice that there is a big jump from numerous instances to a statement which covers all real numbers, or to a geometric proof. It is a matter of debate among historians as to whether, in our terms, the Babylonians had a generalization, and if so whether it was geometric or algebraic or both, or whether they simply had many specific instances.

Pythagoras' Theorem is by no means the end of the story as far as generalization is concerned. Each of the following statements suggests a more general context for looking at the theorem. Some of them suggest a variation on a theme, some suggest a generalization of the theorem, and some are generalizations as stated. You might like to decide which before reading my comments.

A In any triangle, $c^2 = a^2 + b^2 - 2ab\cos(C)$, where a, b, c are the lengths of the sides and C is the angle opposite side c.

B If a regular pentagon is drawn on each edge of a right-angled triangle, then the area of the pentagon on the hypotenuse is the sum of the areas of the other two.

C $3^3 + 4^3 + 5^3 = 6^3$

D The square of the length of the diameter of a cuboid with sides a, b, c is $a^2 + b^2 + c^2$.

E In any triangle, $c^2 = a^2 + b^2 - ab\cos(C)$, where a, b, c are the lengths of the sides and C is the angle opposite side c.

F $2^2 + 6^2 + 9^2 = 11^2$

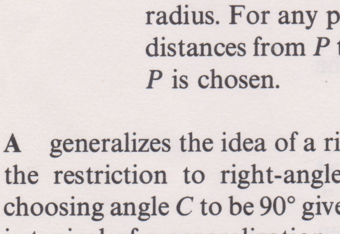
G If x, y and z are positive integers, then $x^n + y^n = z^n$ implies that n must be 2.

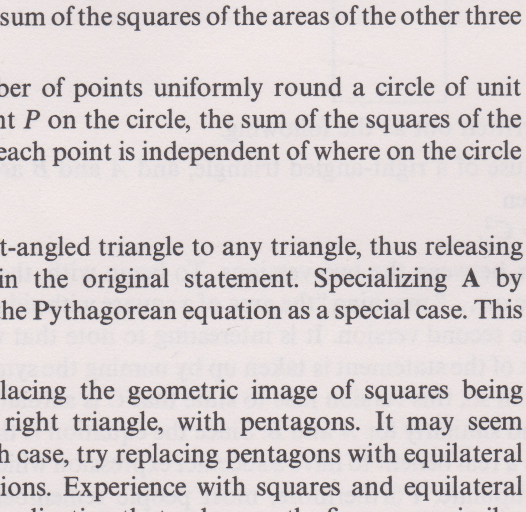
H If, in a tetrahedron, there is a vertex with three right-angles adjacent to it, then the square of the area of the face not containing a right-angle, is equal to the sum of the squares of the areas of the other three faces.

I Place an even number of points uniformly round a circle of unit radius. For any point P on the circle, the sum of the squares of the distances from P to each point is independent of where on the circle P is chosen.

A generalizes the idea of a right-angled triangle to any triangle, thus releasing the restriction to right-angles in the original statement. Specializing **A** by choosing angle C to be 90° gives the Pythagorean equation as a special case. This is typical of a generalization.

B changes the context by replacing the geometric image of squares being constructed on the edges of a right triangle, with pentagons. It may seem implausible at first sight, in which case, try replacing pentagons with equilateral triangles and doing the calculations. Experience with squares and equilateral triangles suggests the further generalization that as long as the figures are similar (that is, scaled versions of the same shape), then the areas add as suggested. This is close in spirit to the way Euclid and his friends are believed to have perceived Pythagoras' Theorem.





C suggests generalizing the idea of adding together squares to adding together cubes, and provides one instance. Adding cubes to get a cube is a variation of Pythagoras' equation rather than a generalization, because it does not specialize back to the original. Neither does it indicate any analogue for the geometrical part of the theorem. Mathematicians have discovered a wealth of interesting questions by following up the numerical aspect while ignoring the geometric, and vice versa.

D generalizes the two-dimensional aspect of Pythagoras to three dimensions. Specializing one of a, b, c to be 0 reduces **D** back to the two-dimensional triangular case.

E generalizes the right-angled triangle notion to other triangles. Specializing by taking angle C to be 90° gives the Pythagorean theorem. *However,* this generalization, while being a reasonable first conjecture, is *not* valid for all triangles, as it stands. For example, it fails for any equilateral triangle (specializing!). It is a generalization, but not a true assertion. Unlike statement **A**, it fails to give correct answers for other values of angle C.

F suggests a generalization from the sum of two squares to the sum of three squares, but offers no geometric side to the analogy. It specializes back to the Pythagorean equation by making one of the squares zero.

G generalizes the idea of integer solutions to the Pythagorean equation to sums of higher powers. This is the famous *Fermat* conjecture, which has so far resisted all attempts to provide a convincing argument that it is true. Who knows, it may be false!

H generalizes the whole Pythagorean theorem to three dimensions, and suggests extending it to higher dimensions still!

I suggests a more general context which glues together several instances of the original theorem (since a pair of diametrically-opposite points forms with P a right-angled triangle). This, in turn, suggests investigating further generalizations—why not an odd number of points; must the points be uniformly spread around the circle; and what happens in three dimensions?

Generalizations lead to conjectures, which may turn out to be true, or false. The first thing to do is to check that they do, indeed, specialize back to your particular cases. Conjectures can then be investigated further by specializing in order to see not *what* might be true, but *why* the conjecture might be true, or *why* it might be false and so need modifying. Suggestions for trying to build convincing arguments to support conjectures are given in the next section.

There are many other general questions which stem from the same Pythagorean Theorem. For example, in order to generate other specializations of

$$A^2 + B^2 = C^2,$$

with A, B, C all integers, it would be tedious to have to explore by trial and error. In fact, it is possible to state a generalization of the three examples

$$3^2 + 4^2 = 5^2,$$
$$5^2 + 12^2 = 13^2,$$
$$8^2 + 15^2 = 17^2,$$

in the form

$(M^2 - N^2)^2 + (2MN)^2 = (M^2 + N^2)^2$ where N and M are any integers.

By specializing this statement, we can recover the particular instances.

TRY $M = 2, N = 1$ and $M = 3, N = 2$ YOURSELF, NOW!

What values of M and N will recover the 8, 15, 17 instance?

(My resolution is on page 58.)

Generalizing

Specializing has demonstrated that the general statement includes the particular instances, but it tells us nothing about whether the generalization is always valid. In this case, it only requires some algebra to demonstrate that

$$(M^2 - N^2)^2 + (2MN)^2 = (M^2 + N^2)^2 \quad \text{where } N \text{ and } M \text{ are any integers.}$$

Expanding each of the terms on the left-hand side we have

$$\begin{aligned} M^4 - 2M^2N^2 + N^4 + 4M^2N^2 &= M^4 + 2M^2N^2 + N^4 \\ &= (M^2 + N^2)^2. \end{aligned}$$

Given that $M^2 - N^2$, $2MN$ and $M^2 + N^2$ provide examples of what are known as Pythagorean triples, i.e. triples of integers which satisfy $X^2 + Y^2 = Z^2$, there are at least three questions which arise by seeking a more general context.

> Can *all* Pythagorean triples be generated by the expressions in M and N?
>
> Can any number appear in a Pythagorean triple?
>
> Can similar expressions be found for generalizations such as the sum of three squares being a square?

The first provides a useful example of what can go wrong when you are trying to produce a convincing argument, so I shall defer considering it until Section 5. The third appears briefly in the exercises at the end of this section. I shall investigate the second. Do not worry unduly about the technical details, because this unit is not about mathematical content. Rather, pay attention to the uses of specializing and generalizing, and the dead-ends encountered along the way, for they typify what it is like to *do* mathematics, as distinct from studying polished mathematics texts.

Can any number appear as one of $M^2 - N^2$, $2MN$ or $M^2 + N^2$? I immediately looked at $M^2 - N^2$, and said to myself that surely any number can be written as the difference of two squares. A tiny bit of specializing showed that my conjecture would need modifying.

Specializing

M	N	$M^2 - N^2$
2	1	3
3	1	8
3	2	5
4	1	15
4	2	12
4	3	7
5	1	24
5	2	21
5	3	16
5	4	9

I do not see how 2 can arise as the difference of two squares. I have obtained 3, 5, 7, and 9, which suggests that I might (being more cautious now!) be able to obtain any odd number. Before diving in further, I pause to recollect what I WANT. I want to know whether any number can appear in some Pythagorean triple. How else might I proceed?

Why not look at the $2MN$ term instead! Given any number, I can factor it, and surely I can choose M and N as two factors. Try 6 for example. I know that $6 = 2 \times 3$, so I can take $M = 2$ and $N = 3$. CHECK! No, I need to have another 2. Try $M = 3$ and $N = 1$. Then the triple will be

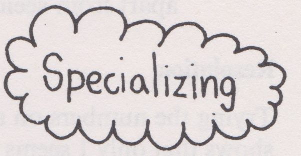
$$3^2 - 1^2, \quad 2 \times 3 \times 1, \quad 3^2 + 1^2,$$

which is

$$8, 6, 10.$$

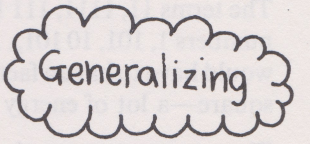
The 2 in $2MN$ is a bit awkward. What shall I do if I am given an odd number? I shall have to resort to the difference of two squares, but now I know that it must be an odd number.

I WANT $2K + 1$ to be written as the difference of two squares. Specializing, using some of the earlier data:



$$1 = 1^2 - 0^2,$$
$$3 = 2^2 - 1^2,$$
$$5 = 3^2 - 2^2,$$
$$7 = 4^2 - 3^2.$$

A pattern is beginning to emerge. Generalizing from this systematic specializing, I try



$$2K + 1 = (K + 1)^2 - K^2.$$

Immediately, I recognize that this is right because I am familiar with the binomial theorem, and I realize that I might have seen it earlier if I had paused for a moment. I have not quite finished however, because I want to exhibit M, N and the Pythagorean triple.

$$M = K + 1 \quad \text{and} \quad N = K,$$

and the triple is

$$2K + 1, \quad 2K(K + 1), \quad 2K^2 + 2K + 1.$$

Have I answered my original question? I have implicitly, but not explicitly. Given any positive integer (I am being careful now), that number can appear as a member of a Pythagorean triple: if the given integer P is even, then M and N can be found by factorizing P as $2NM$, and if P is odd, then M and N can be chosen as shown above.

It is always a good idea to pause when you reach a conclusion, and look back over your work. The details of the argument need to be checked, because it is terribly easy to make an error in a calculation, or to overlook some possibility. It is also important to see what can be learned—for example, in this case I responded too quickly to the first idea that came into my head, and narrowly saved myself from a long diversion considering which numbers can be written as the difference of two squares. It is a worthwhile question to pursue in its own right, but it is not necessary for the original question, as in the end I needed only the special case of representing an odd number as the difference of two squares, which I was able to resolve quite easily. It also occurs to me to investigate which numbers can be expressed as the sum of two squares, and I will certainly need to know that in order to tackle the following.

In how many different Pythagorean triples can a given number appear?

The point is that when one question is resolved, there are usually plenty more arising from it which help to put the original in a broader context.

It is important to realize that, despite the neat and orderly appearance of textbooks, *doing* mathematics entails making plenty of false conjectures, and pursuing many dead-ends. Textbooks omit them because it is not very often instructive to follow other people's dead-ends, indeed it can be intensely frustrating, and surely the purpose of a text is to show how something *can* be done, not how it cannot. In your own studying, and in tackling questions, you will encounter many false trails—do not be disheartened, but rather take pleasure in the hunt, and in the gradual refinement of conjectures. Specialize, preferably systematically, when you reach a stalemate, keeping all the while an eye on the generalization you are after.

Here is another example of a question which invites several different approaches, though most of them are rather hard to follow through!

All Ones

Which of the numbers 1, 11, 111, 1111, 11111, ... can be a perfect square?

TRY IT NOW

STUCK? Try them on your calculator! Make a conjecture. Which digits do perfect squares end in?

Do not spend a long time on this question! It is not worth a lot of effort, apart from seeing the many possible dead or at least difficult "ends".

Resolution

Trying the numbers on a calculator seems a good place to start. The calculator shows that only 1 seems likely to be a perfect square, but it is far from clear why this might be.

The terms 11, 1111, 111 111, ... are divisible by 11, which suggests looking at the numbers 1, 101, 10 101, ... obtained by removing the factor of 11. These numbers would have to have a factor of 11 as well, if the original number was to be a perfect square—a lot of energy can go this way!

The square roots on the calculator are

1, 3.32, 10.54, 33.33, 105.41, 333.33, ...

I could look for a pattern here, trying to make use of the appearance of those threes.

If I stop and ask myself which digits can be the last digits of a perfect square, I find that numbers ending in

1, 2, 3, 4, 5, 6, 7, 8, 9, 0,

when squared, have as their last digits

1, 4, 9, 6, 5, 6, 9, 4, 1, 0.

Only numbers ending in 9 or 1 could possibly be square roots of numbers consisting solely of 1s. Dead halt—what now?

Carry the same idea further—what two-digit numbers can be the last two digits of a perfect square? Careful—that looks like a lot of work. I really want to know whether 11 can be the last two digits of a square. Trying all the cases as I did for single digits *will* get an answer, but so will a little algebra!

Suppose $10A + 9$ or $10A + 1$ were square roots of an all 1s number. Their squares are

$$100A^2 + 180A + 81 \quad \text{and} \quad 100A^2 + 20A + 1.$$

I want the second-last digit to be a 1, but the second-last digit comes from the last digit of $18A + 8$ or from $2A$, which in both cases is even! None of the all 1 numbers can be squares.

Checking over the argument, I wonder where the fact that 1 *is* a perfect square shows up?

Generalizing

I have also shown a great deal more than I wanted originally, because the argument looks at only the last two digits. It never occurred to me at the beginning that only the last two digits mattered—perhaps I was blinded by the plethora of 1s.

The point of the examples is not that they are yet more mathematical content to be learned, but that they illustrate a number of points about specializing and generalizing.

Specializing and generalizing go hand in hand. They cannot easily be separated.

The purpose of specializing is first to give substance and meaning to abstract statements, and then to find out why something might be true, or why it is false. It can be shown to be false by finding a counter-example, and then more specializing is needed to see how to modify the conjecture.

Calculations with specific numbers often suggest what calculations to do with more general symbols.

Setting a result in a more general context can lead to interesting questions, and supports understanding by showing up previously unsuspected connections.

The Pythagorean example also demonstrates how a mathematical investigation might be carried out. Whereas most mathematical texts consist of well worked-over and tidied solutions which frequently have lost all the flavour of the original discoveries, the account given above was just as I worked it out, without editing. Looking back over it you will notice the frequent use of I KNOW and I WANT, which I find useful in order to keep track of where I am and where I am going. Their use will be developed further in later sections.

The next short example illustrates how specializing and generalizing can help each other, by showing that it often helps to do both at the same time when working on a question.

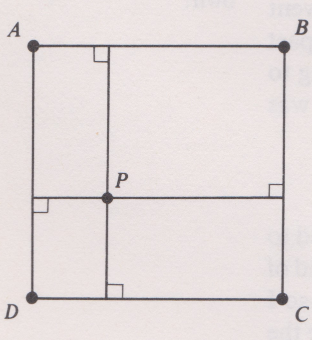
> If P is any point inside a regular N-sided polygon, then the sum of the distances from P to each of the edges is independent of where inside the polygon P is placed.

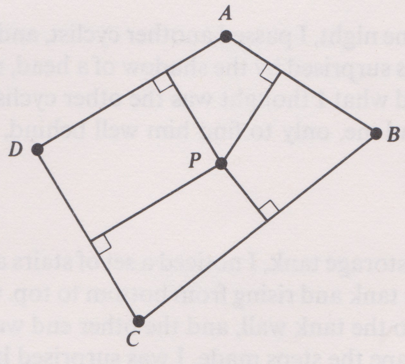
The first thing to do is to specialize.

SPECIALIZE NOW

I am not going to show you my diagrams, because you must draw your own—yours can speak to you whereas mine have to be interpreted and worked on. When specializing, it is sensible to look at cases in which the calculations are reasonably easy, like $N = 4$, $N = 6$, and indeed, as I conjectured and then checked, N even (note the generalizing). Finding a justification for the full claim involves being able to do the calculations in the awkward cases as well, like $N = 3$ and $N = 5$. I twice started some calculations but gave up almost immediately because they seemed so unpleasant.

Before launching in, it is worthwhile considering what is wanted and what is known. In this case, it is known that the polygon must be regular—a rather unsatisfactory state of affairs because I immediately get curious as to why that condition is put in. I tried 3 points (the simplest case), but without the regularity. I couldn't see much, so I tried 4 points without regularity, and it became clear to me that the claim works for the regular case because you can easily do the computation, but as soon as you have irregular figures, the calculations are going to get caught up with the lengths of the sides of the polygon.





That led me to ask how the distances of P to the sides, and the lengths of the sides, might be connected... and suddenly it popped into my head that their product gives the area of a triangle... well, double the area. More specifically (as I looked for a way to record what I had seen), if A and B are two adjacent vertices of the polygon, then the perpendicular from P to AB, times the length AB, is twice the area of the triangle PAB.

In a flash, I saw the original polygon divided up into triangles with P as a vertex common to them all. If the edges of the polygon are all the same length, then the sum of the areas is the sum of the distances from P to the edges, times the constant edge length. Now I could see what the question was getting at.

I can begin to see how I must be a bit careful about whether the polyon is allowed to cross itself, since then areas might overlap, and I have a vague taste of a possible generalization which permits P to be anywhere, not just inside the polygon.

Again, the purpose of this example is to illustrate features of specializing and generalizing, not to carry the investigation to completion. In this case, the act of generalizing at the same time as specializing led me to see an argument that would otherwise have involved fearsome computations.

Specializing and generalizing have, so far, been illustrated as processes for coming to grips with questions or with texts. They actually play an even more fundamental role, for where does the content of mathematics texts come from? The techniques, definitions and results all come as the result of people asking questions, and then trying to answer those questions, however partially. Studying mathematics is not just a matter of studying other people's solutions to other people's questions. Mathematics as a subject comes alive for you when you start noticing your own questions. There are questions around all the time—and the ability to notice them is tied up with seeking generalities. It all has to do with looking at a *particular* situation, and seeing it as representative of a generality. Most questions have the quality of *why* or *is-it-always-so*, which are based on generalizing from the particular.

Whenever you encounter a mathematical or physical situation, asking the questions

what if… is changed?
what happens if… ?
of what is this a special case?

may lead to mathematically rich investigations. "What if" type questions are equally applicable when trying to understand a passage in a text, working on an assignment and even when walking down the street. Used in a text, they can direct attention to other ideas and thereby bring out connections, helping you to place what you are reading in a broader context. Used with an assignment, they may suggest relevant ideas that help you with the problem at hand. Used "out of doors" or "in the market place" they can lead to fascinating and difficult questions. Most of the examples in this unit so far, and in the course as a whole, have come about from asking such questions while paying attention to mathematical ideas. Later in the course, there will be units devoted to solving practical problems, but in the meantime there is no harm in beginning to look out for them yourself. For example, the following.

Do not spend time investigating these now. A better use of time would be to start collecting your own!

Night-rider

Riding on a cycle-path one night, I passed another cyclist, and then went under a street lamp. I was surprised by the shadow of a head, racing past me, so I swerved to avoid what I thought was the other cyclist trying to pass me. I glanced behind me, only to find him well behind. What was going on?

Tanked

Walking near a large oil storage tank, I noticed a set of stairs attached to the side of the cylindrical tank and rising from bottom to top. One end of each step was attached to the tank wall, and the other end was free, so I had a fine view of the shape the steps made. I was surprised because the shape was not what I expected.

Double Glazing

Watching the sun rise one fine morning, I noticed that I could see two suns—a very bright one and a weak one. The weak one appeared just below the bright one, and as I rocked from side to side, the weak one seemed to swing from side to side relative to the bright one. The window was double-glazed, the panes being some 3 inches apart. On another occasion, at a different window, the weak sun appeared to move up and down. Explain!

Umbrella

Walking in a real downpour with an umbrella, I managed to keep my head dry, but little else. How large should an umbrella be in order to be effective, assuming that it is used sensibly?

Freezer

I have noticed that it is very difficult to open the freezer door immediately after having closed it. If I wait a short while, it is much easier. Explain!

See-saw

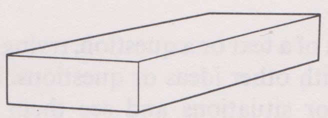
Walking in a playground, I noticed that the see-saws were not like the ones from my childhood, because they remain horizontal when no one is on them. What is the mechanism, and what is the path of the seat as it goes up and down?

It is important to remember that most of the questions generated in this way will be hard—they need to be specialized by simplifying and experimenting, and in most cases the question needs clarifying before it can be worked on. The point is that by *noticing* questions, you exercise your powers of specializing and generalizing.

As a final example of the interplay between specializing and generalizing, consider the following question.

Constant Perimeter

Investigate rectangular boxes in which all faces have the same perimeter.



TRY IT NOW

STUCK? Specialize—make sure you know what the terms mean, then look for an example. Now look for another. Can you detect anything going on? Try to express it in general terms so that it applies to all rectangular boxes.

I found a cube pretty quickly, but each time I tried to produce another example, I couldn't get the third face to have the same perimeter as the other two. (There are six faces, but they are identical in pairs, so really there are only three I need think about.) For example, if the first face is 2 by 3, then it has perimeter 10. The second face has to be 3 by something with the same perimeter, so it must be 3 by 2, but then the third face is 2 by 2 and has the wrong perimeter.

Trying to express in words my sense of what is going on, it came out rather clumsily. Once you have chosen a side and a perimeter, the second side of two faces is determined, and must be equal, so the third face is a square. It cannot have the same perimeter as the other faces unless they are all square. Even I find that hard to follow!

Symbols are much clearer. Suppose the faces of the box are P by Q, Q by R, and R by P. The perimeters are $2(P + Q)$, $2(Q + R)$, and $2(R + P)$, which I know must all be equal. Equating the first two gives

$$2(P + Q) = 2(Q + R),$$

so

$$P = R;$$

and equating the second two gives $Q = P$, so the box must be a cube.

Notice how the succinctness of the algebra communicates so much more clearly than the words—even if you refine the words and say it better. The algebra actually helps the pattern or structure to emerge. As usual, **Constant Perimeter** cannot be permitted to finish there. What other variations are there—what other shapes could I try?

I immediately thought of a tetrahedron.

TRY IT NOW

(My resolution is on page 58.)

One nice thing to emerge from the resolution of the tetrahedron variation is that it leads to yet another conjecture: given any triangle, then a tetrahedron can be constructed with four congruent copies of that triangle as the faces. I invite you to pursue this in the exercises. Specializing with scissors and paper will suggest that this conjecture needs modifying. When a new conjecture arises which seems very plausible, if not downright convincing, how can someone else be convinced? That is the subject of the next section!

Summary

Specializing and generalizing are intimately tied together. Specializing can produce fodder for generalization, and generalizations must be checked to see that they do specialize back to the particular cases which spawned them. Both specializing and generalizing are more subtle than they first appear. Often, it is important to specialize systematically so that a pattern can emerge. At other times, it is helpful to be extreme, to stretch an idea to its limits, in order to see what is going on. The overall purpose is to become aware of, and to enunciate conjectures which can then be examined and either modified or justified—the topic of the next section.

Generalizing also involves reflecting on a section of a text or a question, trying to place it in a broader perspective, to see links with other ideas or questions. By setting yourself to look at particular events or situations and see them as indicating or suggesting a more general phenomenon, you can begin to ask your own mathematical questions. Mathematical thinking may then become alive inside you, and all this "talk" about specializing and generalizing will become second nature.

Exercises

3.1 It has been claimed that if you have four congruent triangles, then these can form the four faces of a tetrahedron. INVESTIGATE!

STUCK? Specialize to concrete materials—scissors and card or paper. Specialize your triangle to be equilateral, then vary it. What sorts of things might go wrong? Try extreme special cases.

3.2 Place the following arithmetic facts in a more general context,

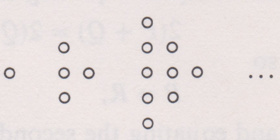
$$1^2 + 2^2 + 2^2 = 3^2, \qquad 4^2 + 4^2 + 7^2 = 9^2,$$
$$2^2 + 6^2 + 9^2 = 11^2, \qquad 4^2 + 6^2 + 12^2 = 14^2,$$

by verifying that for any numbers A, B and C,

$$(A^2 - B^2 - C^2)^2 + (2AB)^2 + (2AC)^2 = (A^2 + B^2 + C^2)^2,$$

and specializing. Show that $2^2 + 3^2 + 6^2 = 7^2$, which is the fourth example given with a common factor removed, and that $3^2 + 4^2 + 12^2 = 13^2$, but that *no* choice of integers for A, B and C will yield these particular cases.

3.3 How many dots will be needed to make the 37th diagram in the following sequence?



Generalize.

STUCK? Specialize by counting the dots, then look for a pattern, then see if the

dots can be rearranged to demonstrate your conjecture. Alternatively, use Exercise 2.2.

3.4 Family Tree

I heard someone on television claim that he was one-third Cherokee. He is in good company, for in the oldest known recorded story, the Sumerian tale of Gilgamesh, it is claimed that Gilgamesh is two-thirds god and one-third man. Investigate these claims.

STUCK? Specialize, perhaps using a family tree. You WANT to find out how such calculations are done, and then how one-third can be an answer.

3.5 Crossless

In the plane, P points are to be joined in pairs by straight-line segments, but they may not cross each other. What is the maximum number of line segments there can be?

STUCK? Reformulate a more precise question for yourself. Specialize. Do not believe your conjectures, explicit or implicit!

Specialize: draw your own diagram.

INTERLUDE C ON EXPECTATION

Studying mathematics can be very frustrating. Some new ideas are encountered, with techniques for solving certain types of problems; there is barely time to get a sense of the ideas before it is time to practise the techniques on some exercises, and then do some assignment questions. Often, it is possible to use the examples and exercises to master the techniques and score good marks on the assignment, yet a few weeks later the ideas and the techniques seem to have evaporated. Even if the techniques are recalled, there may be little satisfaction or sense of understanding. In other words, mastery and understanding may not be quite the same things. I can think I understand something, yet not be able to do the examples; alternatively, I can be able to do exercises perfectly, and not understand much about what I am doing. It is worthwhile, therefore, to clarify expectations about understanding and mastery.

There is one simple lesson that many people find difficult to accept:

I CANNOT expect to MASTER everything on first exposure.

Indeed, it is not often possible to master *anything* on first exposure. It is a bit like being on board ship, pulling into harbour in a fog. A distant foghorn is heard intermittently, but nothing can be seen through the fog. Gradually, the foghorn gets louder, and vague shapes appear in the mist. Finally, the ship pulls up to the dock, and I can get onto firm ground.

The initial foghorn is like initial exposure to ideas—it gives me a sense of direction, a vague notion of what is happening. As I gain experience working on examples, vague shapes emerge from the mist, and I begin to see what is going on. With continued exposure, I begin to discern more details, to see more of what is involved. Eventually, I become aware that it is time to practise the technique or to use the idea in new contexts. With sufficient practice, the techniques and ideas act as firm ground for the growth of understanding. The most important aspect to remember is that it does take time, and that understanding will grow.

In the case of a particular technique, on first exposure, I try to find out WHAT the technique does. With continued exposure to worked examples, I begin to see HOW it works, and begin to get a sense of WHY it might work. In the pressure to keep up with my studying, it is easy to forget that the worked examples are specializations of something, and that the text gives me guidance to the generalization. Thus, any examples I work on should be related directly to the general statements in the text, and not just worked through on their own. Eventually, I reach a point where I recognize that I must get down and master the

technique by doing lots of exercises myself. Trying to do this too early is often a waste of time, and putting it off just clogs up my memory, because once I have really concentrated on mastering the technique, I no longer have to remember it. It becomes second nature.

Sometimes, I find myself short of time, so I use the worked examples and exercises as indicators of how to perform a technique. This is a perfectly mathematical approach, since it is using specializations provided by the text to pick up clues about the general technique. However, too much of this is a waste of my time and mathematical skills, because I am ignoring the text which actually gives guidance as to the generalization. Furthermore, it is all too easy to focus on one salient aspect and miss what the technique is really all about. The result is more often frustration and uncertainty than pleasure and confidence. In the short term, it may get an assignment done, but it does not always lead to understanding.

The pressure of new work is always present when studying mathematics, and it is compounded by the pressure, mounting to hopelessness and panic, of previous work only partly comprehended and insecure. The best way to reduce pressure is to adopt a sensible level of expectation. The pace and type of activity may seem to be dictated by the course materials, and to some extent this is true, but the feeling of being driven forward at breakneck speed is largely due to expectations. If I expect to master everything on first encounter, then I will definitely feel great pressure. If, however, I bear in mind the necessity of a gradual development from first seeing, through increasing exposure, to final mastery, I can relieve a good deal of the pressure. That does not mean that I can sit back and expect the ideas to mature miraculously inside me—I need to keep struggling with the ideas—but being aware of the need for time and repeated exposure can make studying easier, and much more pleasant.

Mathematics is often portrayed as a linear subject that piles up on itself, so that I cannot understand tomorrow's work if I have not mastered yesterday's. In my experience, it is much more of a layering process, of things coming clearer with time and repeated exposure.

Sensible levels of expectation are closely linked to the activities of specializing and generalizing. On first exposure to a mathematical topic, my task is to let go of details and get an overall impression of what is going on. If I am being offered specific examples (specializations), then my task is to generalize, to work on the examples by trying to see what is common, and thus to see what is going on. If I am being offered generalizations or theory at first, then my task is to seek specializations for myself, and this requires specific examples which give me confidence. It is often very tempting to try to work through the theory line by line, but unless this is done at a level of confidence-inspiring manipulable examples, it gives no overall sense of what is going on, and is rarely much use. As I work more closely with the examples, my aim is to try to see for myself the general principle behind them. Each attempt to articulate that generality for myself is a conjecture which is (I hope) leading me towards deeper understanding. When I feel the need to practise the technique on examples, I am laying down a firm bed of practical experience which will come back to me when called upon later.

When written out like this, it seems like a lot to have to do as well as study, but the whole point is that this *is* what studying actually means. It *is* what you are doing already. By being aware of the processes, however, you will almost certainly find that you can become more efficient, and less prey to worries about not mastering things on first exposure.

Advice

Try to establish reasonable expectations of mastery and understanding. Mastery comes with practice, understanding with time.

Do not rush too quickly into trying to master a technique until you have worked on examples and tried to write down for yourself what principles or ideas are being illustrated.

SECTION 4 CONVINCING YOURSELF AND OTHERS

When you think you have understood something, or found an answer to a question, it is easy to be convinced that you are right, simply because it is *your* idea. It can come as quite a shock, several weeks later, to discover that what was once so clear is now obscure or even incorrect. Your idea or solution may be right, but it may be only partly right or even completely wrong. If it is going to serve as a stepping stone in the growth of understanding, then it should be doubted and questioned.

This section is concerned with the transition from convincing yourself to convincing someone else. It is easy to get the impression from mathematics texts that arguments are supposed to spring full blown from someone's mind, when in fact quite the reverse is the case. Any argument of any significance has been through many stages of change and refinement.

The principal device being offered is to pay attention at all times to what you KNOW and what you WANT, both of which are clarified by specializing and generalizing. Every argument (mathematical not social!) consists of a bridge built between what is KNOWN at the beginning, and what is WANTED at the end. However brief or skeletal, the intervening steps are intended to indicate how to pass from the known to the unknown. Despite its simplicity, this observation can be very helpful indeed in trying to resolve mathematical questions.

Consider the following statement, which began the first section, on specializing.

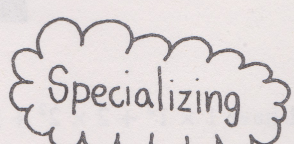
> The sum of the cubes of the first N positive integers is the square of the sum of those integers.

We specialized a few cases to get a sense of what the statement was saying, but did not address the question of whether the statement is actually always correct. In fact, Block II *Unit 3* contains an argument using binomial coefficients. The approach taken here shows another way of looking at it without using the binomial coefficients. It also illustrates how you might go about finding your own argument.

The first thing to decide is whether you are convinced that the statement is reasonable. This means trying some simple examples to see if they are right; but it means more. It means trying to get some sense of *why* it seems right. Just because it checks in three of four cases, we have no reason to believe that it will always work. For example, is it true for the first 37, or the first 137 positive integers? Try some more cases, and try to get a feel for what is happening.

DO SO NOW

After three or four more cases, it becomes hard to believe that it could ever fail! This is the moment to be on guard, because it is not enough to be sure in yourself—as many mathematicians have found to their chagrin. What is needed is a convincing argument, and to find it, you must continue specializing until you have found some reason *why* it works.



In this case, I checked it for $N = 5$:

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

and

$$(1 + 2 + 3 + 4 + 5)^2 = 225.$$

To check it for $N = 6$, I noticed that it is not necessary to repeat all the work:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 &= 225 + 6^3 \\ &= 441 \end{aligned}$$

and

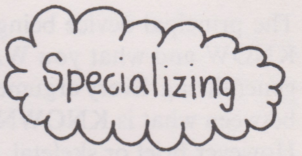
$$\begin{aligned} (1 + 2 + 3 + 4 + 5 + 6)^2 &= (15 + 6)^2 \\ &= 21^2 \\ &= 441. \end{aligned}$$

The current round of specializing is to try to see what is actually involved, and by following the *shape* of the calculations rather than just doing the arithmetic, an idea begins to emerge. Each instance could be based on the preceding case.

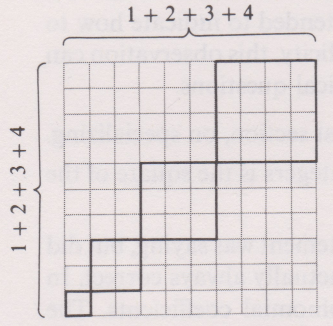
You might be able to see how to exploit the shape of the calculations and write down a general description of how the Nth case follows from the previous case, but my wish at this point was to be able to see directly what was happening, so I looked for a geometric picture.

I WANT $1^3 + 2^3 + 3^3 + \ldots + N^3 = (1 + 2 + 3 + \ldots + N)^2$.

I KNOW very little. I have no feeling for what sums of cubes are like, but I do know that squares can be pictured as squares on paper, so I am going to go for a diagram. I want to picture N^3. A cube seems an obvious idea, but does not appeal to me because I cannot see how to pile them all up. All I can think of is N copies of an N by N square, which at least I can draw. My overall idea is to draw a diagram which shows the growth of the sum of the cubes and the square of the sum, as N increases.

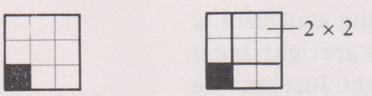


I begin by drawing a series of squares of size 1 by 1, 2 by 2, 3 by 3, … I want them to pack into a square corresponding to the square of the sum, of size $1 + 2 + \ldots$.



Try some cases.

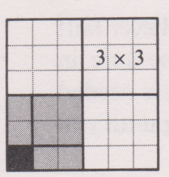
$N = 2$. Want Try



AHA! The rectangles are each half a 2 by 2 square.

This is a diagrammatic version of $1^3 + 2^3 = (1 + 2)^2$.

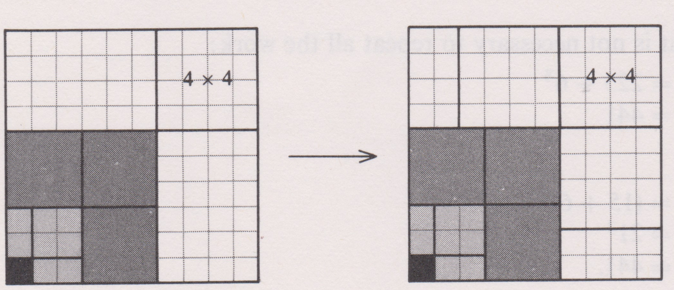
Try $N = 3$.



AHA! My three 3 by 3 squares appear.

I have $1 \times 1^2 + 2 \times 2^2 + 3 \times 3^2 = (1 + 2 + 3)^2$.

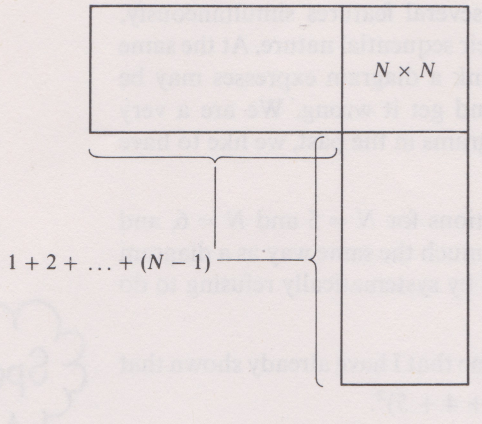
Try $N = 4$.



I need look only at the L-shape added on to the previous case. The two rectangles make the fourth square.

I feel ready to try a more general case now.

I KNOW that I have to fill an L-shape which is N steps wide and which has arms of length $1 + 2 + 3 + \ldots + (N - 1)$.



An N by N square fits in the corner. How many N by N squares fit in each arm? I expect different answers for N even and N odd.

To know the lengths of the arms, I need to know $1 + 2 + 3 + \ldots + (N - 1)$. It is $N(N - 1)/2$. So the two arms together constitute a strip which is $N(N - 1)$ steps long and N steps wide. Such a strip will be filled by $N - 1$ squares which are N by N, so including the one at the corner, there are N squares of size N by N to fill in the L-shape, exactly as requested. When N is even, one N by N square will appear as two rectangles.

The important feature of this example is *not* in the details of the argument, but in the helpfulness of referring frequently to what is KNOWN and what is WANTED. Initially, these are provided in the question. As work begins, more specific and detailed WANTS arise, and at the same time new relevant KNOWNS are generated.

Whenever I get stuck trying to produce a convincing argument, I find it helpful to write down everything I know that seems relevant—it rarely takes me long, and it helps to get it on paper in front of me, because otherwise my head gets too full of half-formed ideas to be able to concentrate properly.

The argument as written down is just as it occurred to me. I find it convincing because it shows how the sum of cubes and the square of the sum grow with increasing N. I no longer depend on a few examples, nor on a vague feeling that it seems to be right. I can actually see the structure in the diagrams. It is almost as if the diagrams speak—but it takes a lot of words to write out what they say. To make it convincing to others, I shall have to add more words, structured by KNOW and WANT. Why not try right now to write down in words and symbols what the diagrams are actually saying?

WRITE IT DOWN NOW

Here is my version.

Assuming I KNOW $1^3 + 2^3 + \ldots + (N - 1)^3 = (1 + 2 + \ldots + (N - 1))^2$ for a particular value of N, I WANT to add N^3 to both sides of the equality and see if the right-hand side is the square of the sum of the integers from 1 to N. The diagram shows me that N^3 is the area adjoined, which is 2 copies of $\dfrac{N(N-1)}{2} \times N$ plus N^2 in the corner.

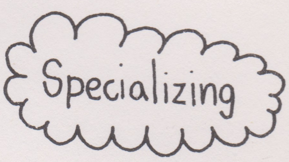
The right-hand side is now

$$(1 + 2 + \ldots + (N - 1))^2 + 2N^2(N - 1)/2 + N^2$$
$$= (1 + 2 + \ldots + (N - 1))^2 + 2(1 + 2 + \ldots + (N - 1))N + N^2.$$
$$= (1 + 2 + \ldots + N)^2.$$

This is what I WANTED.

It is not always easy to write down what you see in a diagram, which is why diagrams are so useful. The eye can take in several features simultaneously, whereas words are often limiting by virtue of their sequential nature. At the same time, diagrams have a weakness. What we think a diagram expresses may be sufficiently complicated that we are misled, and get it wrong. We are a very literate culture, and having been misled by diagrams in the past, we like to have everything written out in symbols.

Earlier, I wrote down the arithmetical calculations for $N = 5$ and $N = 6$, and suggested that these might also speak to you in much the same way as a diagram does. In this case, an argument can be revealed by systematically refusing to do arithmetic calculations.



Let us repeat the $N = 5$ and $N = 6$ cases. I assume that I have already shown that

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2.$$

I WANT

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = (1 + 2 + 3 + 4 + 5 + 6)^2.$$

Using what I KNOW in the left-hand side,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = (1 + 2 + 3 + 4 + 5)^2 + 6^3.$$

I WANT this to be the same as

$$(1 + 2 + 3 + 4 + 5 + 6)^2,$$

which I KNOW to be the same as

$$(1 + 2 + 3 + 4 + 5)^2 + 2(1 + 2 + 3 + 4 + 5)6 + 6^2.$$

Comparing the two expressions, I find that $(1 + 2 + 3 + 4 + 5)^2$ is common to both sides, so I WANT to work on the remainder.

The awkward term is $(1 + 2 + 3 + 4 + 5)$, which I KNOW to be $5 \times 6/2$, using the formula for summing consecutive integers.

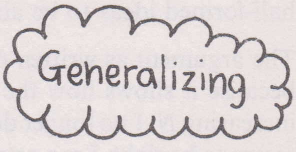
I WANT the value of $2(1 + 2 + 3 + 4 + 5)6$ which is

$$2 \times 5 \times 6 \times 6/2 = 5 \times 6 \times 6.$$

To this I must add 6^2, giving

$$5 \times 6^2 + 6^2 = 6^2(5 + 1)$$
$$= 6^3, \quad \text{as requested.}$$

Now I can go through and replace 5 by $N - 1$ and 6 by N. The result of this act of generalizing will be the same as my description of the diagram.



TRY IT NOW

One further point which emerges from the sum of cubes is the difference between convincing yourself and convincing someone else. It is so easy to be beguiled by a conjecture, especially if it is your own. It is essential to get it written down, and then to disbelieve it. This means more specializing, in a more detached way, trying to see *why* it might be right, or how some example might show it to be false. When you have convinced yourself, then it is time to convince someone else, either real or imagined. On this next run through, you must be extremely sceptical, just as a colleague or tutor might be. You look for places where there is still a gap between a detailed WANT and a detailed KNOWN, and see if it needs bridging with more detail. The best way to learn to be sceptical is to practise on other people's arguments.

To tackle someone else's argument, you first need to get an overview of what the argument purports to demonstrate. Then, line by line, you ask yourself

What do I KNOW?

and

What do I WANT?

in that line. Can a bridge be built, or is there a difficulty? When you get to the end, it is worth reviewing the argument to pick out the main features, the turning points, for the rest of it can be reconstructed from them.

Here is a short example from the text of Block II *Unit 2*.

> The technique ... can be adapted to solve any equation of the form
>
> $$a^x = k,$$
>
> for a, k positive real numbers and $a \neq 1$.
>
> Specifically, the technique is as follows.
>
> $$a^x = k.$$
>
> Apply $\log_{10}$:
>
> $$\log_{10}(a^x) = \log_{10}(k),$$
>
> then
>
> $x\log_{10}(a) = \log_{10}(k)$ ("log of a power = multiple of the log"),
>
> then
>
> $x = \log_{10}(k)/\log_{10}(a)$ (because $a \neq 1$, we know that $\log_{10}(a) \neq 0$).

When KNOW and WANT are inserted, it comes out as follows.

I WANT to find x explicitly, given $a^x = k$.

I KNOW $a^x = k$, $a \neq 1$.	I WANT x; I WANT to get x out of the index position.
I KNOW logs strip down exponents.	I WANT to apply logs to both sides.
I KNOW $\log_{10}(a^x) = \log_{10}(k)$.	I WANT x.
I KNOW $\log_{10}(a^x) = x\log_{10}(a)$ (that's why I used logs!).	I WANT x.
I KNOW $x\log_{10}(a) = \log_{10}(a^x)$ $= \log_{10}k$.	I WANT x.
I KNOW $x = \log_{10}(k)/\log_{10}(a)$.	Which is what I WANTED!

Having inserted all the KNOWs and WANTs, I found no lines which needed augmenting. The advantage of putting them all in is that at each stage I am reminded of where I am going—one of the hardest parts of reading mathematics is retaining a sense of where you are going, in the midst of algebraic detail.

At the end of this text extract, it says that the technique is straightforward, and it is probably easier to apply the technique from first principles each time than to try to remember the formula. What is involved in remembering the technique? Looking back over the argument, the key features are the use of logarithms to strip off the exponential, and keeping track of what you want, namely to solve for x.

The argument can now be collapsed to

> solve $a^x = k$ for x,
> take logs of both sides, and
> $x = \log_{10}(k)/\log_{10}(a)$.

This very truncated version can be expanded using KNOW and WANT, whenever this is needed.

Here now is an example for you to try, with my resolution, which you should look at only after you have made an attempt. It is too easy to read my version and then

say afterwards, "oh yes, I could have done that", or even "I could never do that!". The only way to strengthen your mathematical muscles is to exercise them, on problems as well as texts.

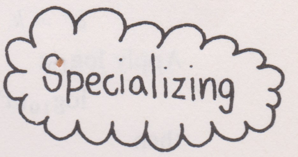
Divisors

Is there a number with exactly fifteen divisors?

TRY IT NOW

Resolution

Fifteen is far too large. What shall I do? I could flail around looking for a number that works; I could simplify 15 and look for numbers with just 1 divisor, then 2, then...; I could look at the number of divisors of 1, 2, 3, 4,... in turn. I shall specialize down to 1, 2... divisors. I must be clear about what divisors are, so try some examples,



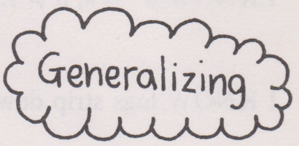
1 has 1 divisor, namely 1.
2 has 2 divisors, namely 1 and 2.
3 has 2 divisors, namely 1 and 3.
4 has 3 divisors, namely 1, 2 and 4.

I wrote down the number of divisors of 5, 6, 7, 8, 9, 10, 11 and 12, and I recommend you do too. I went as far as 12, because I know 12 has more divisors than the earlier numbers.

DO SO NOW

Now, what do I WANT?

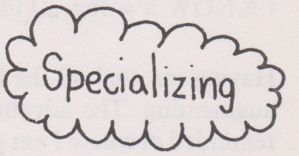
I want a number with 15 divisors. It is going to take a long time to build up to that, so how can I see what is really going on? What do I KNOW about divisors?



If I am looking at divisors of N, then they come in pairs, because if a divides N, so does N/a. OOPS! If N is a perfect square, then its square root has no mate. But that is the only way that a divisor would not be paired up. I have reached a pretty strong conjecture.

CONJECTURE: only perfect squares have an odd number of divisors.

I checked it on the examples I had already done. It seems right in those particular cases, and my argument seems OK. I am pretty satisfied with the conjecture, so I shall carry on. I WANT a number with 15 divisors, so it must be a perfect square. Because it is a perfect square, one of the divisors is accounted for, but where will the other 14 come from? More systematic specializing is needed with perfect squares!



1 has 1 divisor.
4 has 3 divisors.
9 has 3 divisors.
16 has 5 divisors.
25 has 3 divisors.
36 has 9 divisors.

Notice that the specializing is slightly different now, as a result of the conjecture. What about 7 divisors? Can that ever happen? Try some more:

49 has 3 divisors.
64 has 7 divisors.

Now I begin to see a pattern. Look at the divisors of 64 clustered either side of 8:

1	64
2	32
4	16
8	

I conjecture that to get 15 divisors, I should take 2 raised to the $(15 - 1)$, since one divisor comes from the square root, and the other divisors are powers of 2, coming in pairs.

CHECKING: The divisors of 2^{14} are

$1, 2, 2^2, \ldots, 2^{11}, 2^{12}, 2^{13}, 2^{14}$.

It is important to reflect on what has happened, so that I can learn from the experience. My immediate feeling on writing down the powers of 2 was that somehow I "ought" to have seen that right away. What seems obvious now was not obvious when I started! I have accomplished a good deal more than was asked, however, so it is useful to be clear on just what I have found out! At the same time, I want to write down a convincing argument. All that I have written so far is a record of my thoughts. Now I massage it into presentable mathematics!

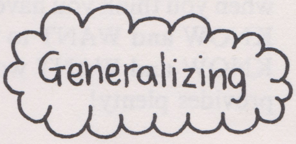
Conjecture 1: A number N has an odd number of divisors if, and only if, it is a perfect square.

Argument: If I KNOW that N is a perfect square, then all of its divisors come in pairs except for the square root of N, so N has an odd number of divisors. If I KNOW that N has an odd number of divisors, then when the divisors are paired up, there must be one divisor left over which pairs with itself. It must be the square root of N, so N is indeed a perfect square.

Conjecture 2: For any odd number N, there is a number with exactly N divisors.

Argument: $2^{(N-1)}$ has exactly N divisors.

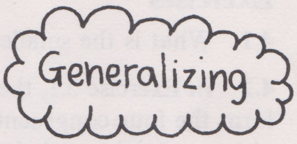
I have not used the fact that I KNOW N is odd, anywhere in the argument. This suggests the following.



Conjecture 3: For any number N, there is a number with exactly N divisors.

Argument: $2^{(N-1)}$ has exactly N divisors.

What is the special role of 2 here?



Conjecture 4: There are infinitely many numbers with exactly N divisors.

Argument: For any prime number p, $p^{(N-1)}$ has N divisors.

Now I want to look at a general number and see how many divisors it has. For example, what happens if I take two primes, p and q, and form

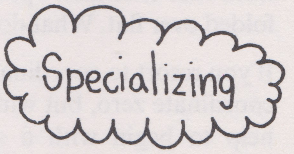
$p^{(N-1)}q^{(M-1)}$? How many divisors does it have?

TRY IT NOW

There are N choices for powers of p, and M choices for powers of q, so there are NM choices all together. So I could get 15 divisors from any number of the form

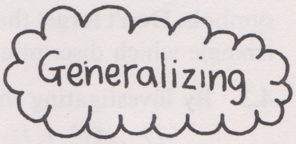
$p^{3-1} \times q^{5-1}$.

Taking $p = 3$, $q = 2$ gives me



$$\begin{aligned} 3^2 \times 2^4 &= 9 \times 16 \\ &= 144, \end{aligned}$$

which is considerably smaller than $2^{14} = 16384$. I am struck by how much more helpful the notation of $3^2 \times 2^4$ is than the decimal notation 144.



Now I find myself posing the new question: what is the smallest number with exactly N divisors? Which is likely to be smaller, $3^2 \times 2^4$ or $2^2 \times 3^4$? What does this suggest in general?

TRY IT NOW

I have an outline idea—factor N and use a generalization of the $p^2 \times q^4$ idea.

What is the value in tackling a question like **Divisors**? I suggest that unless you pause and reflect on what actually happened, there is very little value at all! Now that you are familiar with the mathematical content of the question, you can look back over what happened and learn from the experience. I suggest that, in particular, you review the resolution above, paying particular attention to the

way in which frequent clarification of KNOW and WANT enabled a bridge to be seen between them.

REVIEW IT NOW

It is often the case that as you specialize, what you KNOW actually changes, and bearing constantly in mind what you WANT brings about subsidiary aims or new WANTs. Frequently, students and researchers become so caught up in their specializing and in what they KNOW, that they lose sight of what it is they really WANT. That is why I have been stressing both KNOW and WANT at the same time. By keeping a balance between them, devoting some attention to each, a sense of direction emerges. Furthermore, any resolution will consist of a bridge between KNOW and WANT. The act of setting down a convincing argument involves retracing your steps over all the intervening KNOWs and WANTs, constructing a sequence of little bridges, rather like a medieval beam bridge.

Summary

A convincing argument consists of a bridge built between what is KNOWN at the beginning, and what is WANTED. An incomplete or confusing passage in a text can be clarified by setting out what is known and wanted at each stage. An investigation can be directed by paying attention to what is known and wanted, and by constantly modifying or rephrasing them, looking for a bridge. Finally, when you think you have a convincing argument, it can be structured in terms of KNOW and WANT to help your reader see all the steps. The only way to get KNOW and WANT working efficiently for you, is practice—the next section provides plenty!

Exercises

4.1 What is the smallest number with exactly 12 divisors?

4.2 In Exercise 3.1, there was a conjecture that any acute-angled triangle can form the four congruent faces of a tetrahedron, but that it does not work for obtuse or right-angled triangles. Scissors and paper provided evidence, but a convincing argument is needed. Construct one!

STUCK? Somehow, you have to get a grip on a general triangle, and literally construct a tetrahedron, imposing the condition that all the faces need to be congruent. A three-dimensional object is hard to work with, so coordinates might suggest themselves—but they must be carefully chosen. Before resorting to coordinates, note that you WANT something about obtuse and acute angles, so focus on angle sizes. You KNOW that the tetrahedron can be folded together from four triangles. Specialize to the case when two of the flaps just meet when folded over flat. What do you KNOW about flaps that just meet, or fail to meet?

If you resort to coordinates, try putting a general triangle in the plane with third coordinate zero, but with as many other coordinates zero as possible. It might help to begin with a specific numerically-coordinated triangle, and try to construct the tetrahedron. This will indicate what calculations to do with symbols. Don't forget that you WANT to end up with a condition on the original triangle which discriminates between obtuse- and acute-angled triangles.

4.3 By investigating the sequence of numbers

$$\sqrt{2},\ \sqrt{2}^{\sqrt{2}},\ (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}},\ldots$$

show that it is possible to find an irrational number which can be raised to an irrational power, and yet yield a rational number as the result. Generalize.

4.4 Given any two rational numbers, can you find a rational number in between them which has a power of ten as its denominator?

Given any two irrational numbers, can you find a rational number in between?

Given any two rational numbers, can you find an irrational number in between?

STUCK? What is a rational number? What is an irrational number? Try some examples. How can rationals best be represented—as fractions, decimals, ...?

4.5 If $\frac{A}{B} < \frac{C}{D}$, what can be said about the ratios $\frac{A+C}{B+D}$ and $\frac{2A+3C}{2B+3D}$?

Place this question in a more general context by constructing a story about a journey, with different average speeds during different phases. Place it in a more general context by drawing a parallelogram with vertices at the origin, the points (B, A) and (D, C), and the point $(B+D, A+C)$.

STUCK? Specialize, then generalize.

INTERLUDE D ON BUILDING CONFIDENCE

Lack of confidence is one of the main features of being a student. For me, very often it turned out that I actually knew a great deal more than I thought, but I lacked confidence in myself, and in my ability to get started on a question. What is the basis of mathematical confidence?

Here is a list of increasingly abstract mathematical ideas, at least some of which you have met before:

the integer 37
the rational number $\frac{4}{5}$
the real number $\sqrt{2}$
the real number π
the real number e
the real number x
the point (x, y) in the plane
$\{(x, y): y = 3x + 2\}$
$\{(x, y): y = mx + c\}$
the function $x \longmapsto x^2$
the function $\sin: \mathbb{R} \longmapsto \mathbb{R}$
the function $f: \mathbb{R} \longmapsto \mathbb{R}$
$\{(x, y): y = f(x)\}$
$\{F: F \text{ a function } \mathbb{R} \longmapsto \mathbb{R} \text{ and } F(0) = 0\}$.

I expect that you feel confident that you can manipulate some of these, but not others. For some, you will have a sense of what they are about, while others may be more or less mysterious. But how many of them would have inspired your confidence before you began this course? I suspect that many more would have belonged to the nonsense category than do now. You *have* made progress!

Starting with manipulating objects, which may be physical objects, or more abstract mathematical gadgets like the ones listed, I gradually develop a sense of a more complex notion, and then try to capture it in some sort of picture, words and notation. With experience and practice, the symbols become more succinct, and at the same time more confidence-inspiring. With mastery, they turn into confidence-inspiring objects which I can use in further specializations to build yet more complex notions, such as those near the bottom of the list.

One of the characteristics of mathematical thinking is the number of times I traverse a spiral from

confidently-manipulable objects

through

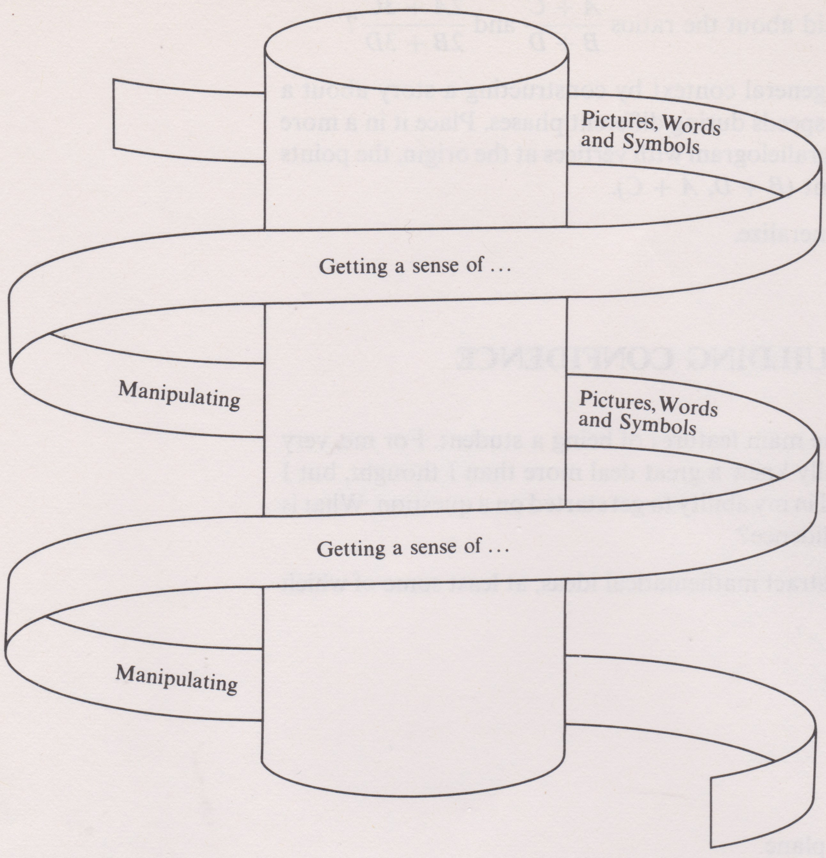
a sense of a general notion common to the specializations

through

an increasingly succinct notation for the generality, in pictures, words and symbols

to

new confidently-manipulable objects.



This spiralling goes on and on in mathematics, giving rise to the frequently misunderstood idea that mathematics is a subject in which you cannot understand a new idea unless you have understood everything that has gone before. Instead of an all-or-nothing "have you got it or haven't you", my experience is much more like layers of earth being laid one on top of another, ultimately providing soil for germinating new seeds. What at first seems highly abstract will, with time and experience, become familiar and confidence-inspiring. The particular collection of objects which I can confidently manipulate is my base. Yours is probably different. We can both tackle a new idea, but we shall use different examples on which to specialize. Consequently, the way we speak about what is going on is likely to be different, at least at first.

Transition from being able to manipulate examples to having a sense of what is actually going on involves specializing and generalizing, which the main sections of this unit have described in detail. Moving from a vague sense of something which is pre-verbal, to being able to capture that "sense of", involves taking time over trying to say what it is that I see, and then recording it in words, in mixtures of words and symbols, and finally in succinct symbols. Even that is not the end of the story, however, because the symbolic version is of no particular value unless it speaks to me directly, and unless I can confidently manipulate those symbols. To reach this state takes time and practice. Instead of rushing into symbols, it is very sensible to stick to versions that I do feel happy with. This often means translating other people's symbols into my own words and pictures, and that is precisely how any mathematician deals with new material. Put another way, it is virtually impossible to *read* mathematics—it has to be done, to be worked through with pencil and paper.

Being aware of the spiral of

manipulating,
getting a sense of,
capturing in symbols,
fodder for further manipulation,
. . .,

can assist in keeping expectations at a reasonable level, and in framing clearly the immediate study task and correspondingly appropriate activities. Students who expect to jump from exposure to mastery are liable to miss out the essential intermediate stage of transforming experience with special cases into their own generalizations. The whole purpose for recommending that you spend time

specializing	—manipulating confidence inspiring objects —trying to get a sense of a pattern or idea
generalizing	—trying to talk about that "sense of" —trying to capture the pattern or idea succinctly
exercising	—practising examples and techniques to reach mastery so that the succinct articulation becomes in its own time a confidently-manipulable object

is so that when you encounter a difficulty, you can track back down the spiral. You will then have a variety of written, pictorial and verbal descriptions which contribute to your growing sense of the topic at hand. Because *you* have articulated them, they are yours, and they can be accessed. If they simply come from a book, then they are somebody else's!

Advice

Abstract symbols will become concrete and confidence-inspiring if they are succinct summaries of ideas that you are confident with. Don't rush into using symbols, but don't be afraid of them.

When a text or question seems overwhelming in its use of symbols or complex terms, don't panic! Backtrack, rewriting them in a form that gives you confidence—specializing—and then work your way back to the substance of the passage—generalizing.

At the end of each study session, and each text section, write down your own summary of what it was about—the main ideas, the techniques, and the sticky places. Try rehearsing them to yourself while waiting for a bus or at other odd moments during the day. If possible, compare notes with other students, or with your tutor.

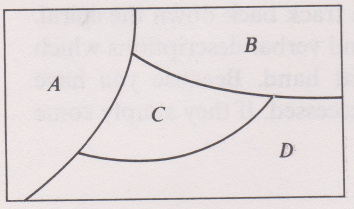
SECTION 5 WHEN IS AN ARGUMENT VALID?

Since doing mathematics is concerned with investigating generalizations and convincing both yourself and others that the generalizations are valid, it would be nice to be able to say definitively what constitutes a valid and totally convincing argument. Unfortunately, this is rather difficult. In order to be mathematically precise, it would be necessary to convert the proffered argument into long cumbersome strings of symbols, with minute details justifying every little step. Mathematicians (apart from those interested in the theory of proofs) are much too interested in the investigation itself to be bothered with all the fine details which only get in the way and obscure the overall picture, so the detailed steps are omitted. Usually this is perfectly safe—but occasionally the steps become leaps over unnoticed chasms. Indeed, it has been suggested that despite rigorous refereeing of mathematical journals, most research articles contain errors, and a high proportion contain serious errors or omissions. These have a habit of turning up later, sometimes much later. Many of these errors can be attributed to the high costs of paper, and hence the demand for succinct outlines of proofs, but it also happens that subtleties are overlooked. If mathematicians have trouble with their arguments, it is not surprising that students do too.

This section outlines a few examples of oversights which have been made in the past, but which have later led to fruitful mathematical investigations. The bulk of the section is devoted to examples of partial and erroneous arguments for you to examine critically.

Famous Oversights

The now famous *Four-colour Theorem* says that in any map on a flat piece of paper, the countries can be coloured using only four colours, in such a way that no two countries with common boundaries have the same colour. The first "proof" appeared in 1879, and was accepted for some 11 years as valid, before being shown to have a missing step that no one could bridge. The missing step turned up when someone tried to generalize the argument to maps on different surfaces, like an inner-tube, and found that the method could be adapted to work in all cases *except* the plane! It was only in 1976 that a convincing argument was found for the planar case, and that involved many hours of computer calculations.



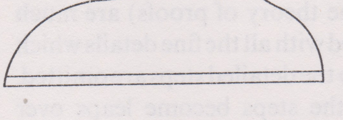
One point that arises from this example is that a general statement can be shown to be inadequate by exhibiting just *one* case, one specialization which is false—often referred to as a counter-example. Thus the statement

> all primes are odd,

is shown to be false by observing that 2 is not an odd prime. In this case, it is the *only* counter-example! To show that an argument is not valid can be a lot trickier. If the statement it purports to justify is itself false, then of course the argument must be false. In the case of a result that turns out to be true, such as the four-colour theorem, errors in arguments cannot be found by producing counter-examples. The place where an unjustified leap occurs must be found. Mathematics departments in universities frequently receive documents purporting to prove famous outstanding conjectures, and each must be seriously read, because they might conceivably be correct. By way of contrast, arguments that purport to show how to trisect an angle using ruler and compasses only, are known to be in error because what they try to prove has already been shown to be false.

Another interesting historical example is given by *Steiner's* famous argument to justify the statement

> the largest region which can be surrounded by a stick, with a rope whose ends are tied to the ends of the stick, is a segment of a circle.



The thrust of the argument is that if the stick and the rope do not form a segment, then a construction can be given for encompassing a slightly larger region. The only configuration which can *not* be made larger by this construction is a segment, therefore a segment of a circle gives the largest area.

The details of the construction are not important, for the subtlety lies in the assumption that because the area can be made larger and larger, it must converge to the region which cannot be made larger by this construction. Intuitively, it seems clear that the area encompassed cannot grow indefinitely large, but this needs checking and saying, even though it does turn out to be valid. Nevertheless, there is a similar argument which indicates what might go wrong.

CONJECTURE: One is the largest whole number.

Argument: Any whole number can be made larger by squaring it, except for 1 which stays the same. By the same argument as in *Steiner's* semi-circle, keep on making the number bigger by squaring it, and since 1 is the only number that stays fixed by this construction, you must eventually get close to 1. Thus 1 must be the largest.

Steiner's argument *is* valid, whereas this is not, but the same strategy is used. Where does the analogy break down?

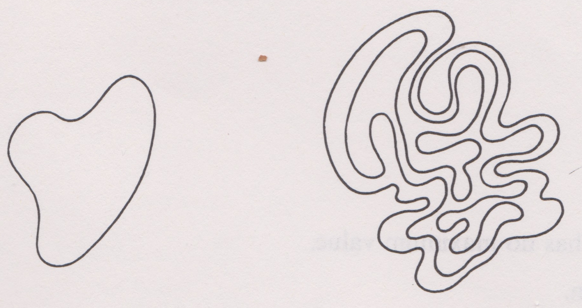
TRY IT NOW

(My resolution is on page 65.)

Intuition about when things converge needs to be developed and tested, because our naive intuition is not always trustworthy. Surprises and unexpected difficulties should all be treated as opportunities to extend and correct your sense of how things fit together. The definitions and theorems of mathematics are aids to refine and develop intuition, not things to be memorized.

Very often, the existence of an undetected subtlety emerges when someone appears to be excessively pedantic in their questioning of hidden assumptions, and as a result, new areas are opened for mathematical investigation. Here are two examples.

It seems quite clear that if you pick up a pencil and draw a continuous curve which never intersects itself, finally ending up by joining on to where you started, then what you have done is to divide the plane piece of paper into two regions—an inside and an outside.



Unfortunately, it is not always clear in a complicated drawing just which is the inside and which the outside. When you try to write down an argument to justify this seemingly innocent and "obvious" assumption, it turns out to be harder than it looks. Indeed, it requires a great deal of mathematical machinery in order to give a convincing proof, because lying behind the difficulties are several unstated assumptions about what constitutes a plane, and a continuous curve. Investigation of this question, once it was exposed as being not completely obvious, opened the way to the whole subject of topology, which lies at the heart of many modern approaches to mathematics.

If I have a drawer full of pairs of socks, it seems intuitively reasonable that I could select one sock from each pair and put them into another drawer. If I imagine a drawer with an infinite number of socks, it seems reasonable that I could still select one sock from each pair. Our intuition with finite sets of objects does not always carry over accurately to infinite sets, however. The whole question of what constitutes an infinite set raises philosophical difficulties which took mathematicians a long time to deal with. For example, Bertrand Russell suggested that since some words, like "short" also describe themselves, but others, like "monosyllabic" do not describe themselves, the word "heterological" could be defined as an adjective describing the property of a word to fail to describe itself. The question then arises as to whether "heterological" is itself heterological, that is whether it describes itself.

Which ever way you decide, you reach a contradiction. Thus if "heterological" is itself heterological, then it fails to describe itself. But "failing to describe itself" is exactly what heterological means, so it does describe itself—a contradiction! Investigate the result of taking "heterological" as not being heterological.

TRY IT NOW

Our intuitive ideas about forming sets and making choices break down in language and in mathematics, until they are refined, and made more mathematically precise.

Looking for Omissions

If great mathematicians overlook steps in their arguments, what hope is there for lesser mortals? The short answer is that there is no foolproof antidote, but there are steps to be taken to exercise care, and to improve your mathematical thinking. They are the following.

1. Always be clear about the status of a statement—conjectured or proved. Remember that specialization is suggestive, and can *indicate* why something might be true, but does not constitute a proof unless all possible specializations have been tested.
2. Always state clearly what you are aware of assuming. One good way to do this is to state clearly what you KNOW at each stage.
3. Keep clear what you KNOW and what you WANT. If you modify what you WANT, remember to check later whether you have really answered the original question.

The remainder of this section consists of examples of valid and invalid arguments for you to examine critically.

Examples

(My comments begin on page 65.)

5.1 The function $x \longmapsto \log(\log(\sin(x)))$ has no maximum value.

Argument: log is an increasing function.

5.2 The function $x \longmapsto \sin^{-1}(2 + x^2)$ never exceeds π.

Argument: The inverse function of $x \longmapsto \sin x$ never exceeds π.

5.3 If $(x - 2)^2 + (y + 3)^2 = 0$, then $x = 2$ and $y = -3$.

Argument: The square of a real number is never negative.

5.4 The function $F: x \longmapsto x^3 + 2x + 1$ has no zeros when x is an integer and all calculations are done in arithmetic modulo 3.

Argument: $F(0) = 1$, $F(1) = 1$, $F(2) = 1$.

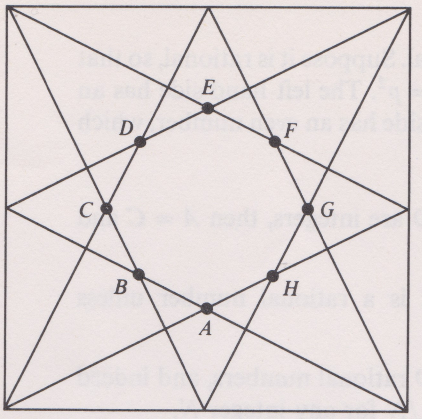
5.5 In a certain university, there are 14 times as many students as professors. The equation which represents this is $14S = P$.

Argument: Translate the phrase 14 times as many students as professors, word by word.

5.6 The number $2 \times \pi^{\sqrt{163}}$ is an integer.

Argument: My calculator gives it as 4 448 511.

5.7 The inner polygon in the figure at the top of page 47 is a regular octagon.



Argument: The figure is symmetrical about the line AE, so AB is equal to AH. The figure remains the same if rotated through 90°, so all edges of the inner polygon are equal.

5.8 If A, B, X and Y are positive real numbers, then

$$\sqrt{A+B} \times \sqrt{X+Y} \geq \sqrt{AX} + \sqrt{BY}.$$

Argument: Square both sides to get

$$\begin{aligned}
(A+B)(X+Y) &\geq AX + BY + 2\sqrt{AXBY}\\
AX + BX + AY + BY &\geq AX + BY + 2\sqrt{AXBY}\\
AY + BX &\geq 2\sqrt{AXBY}\\
(AY+BX)^2 &\geq 4(AXBY)\\
A^2Y^2 + 2AYBX + B^2X^2 &\geq 4AXBY\\
A^2Y^2 - 2AYBX + B^2X^2 &\geq 0\\
(AY-BX)^2 &\geq 0, \quad \text{which is true.}
\end{aligned}$$

5.9 If you take K consecutive numbers and add them, then exactly one of the K consecutive numbers *or* their sum is divisible by $K + 1$.

Argument: For $K = 2$ it is clear, because if the two consecutive numbers are N and $N + 1$, then their sum is $2N + 1$. Then N, $N + 1$ or $2N + 1$, is divisible by three, but in each case neither of the others can be. The argument for $K > 2$ proceeds in the same fashion.

5.10 There is no difference between tall men and short men.

Argument: A seven-foot man is certainly tall, and a four-foot man is certainly short. There is no visible difference between men who differ by 0.1 of an inch. Therefore there is no difference

between men who are 4 feet, and men who are 4 feet 0.1 inch;

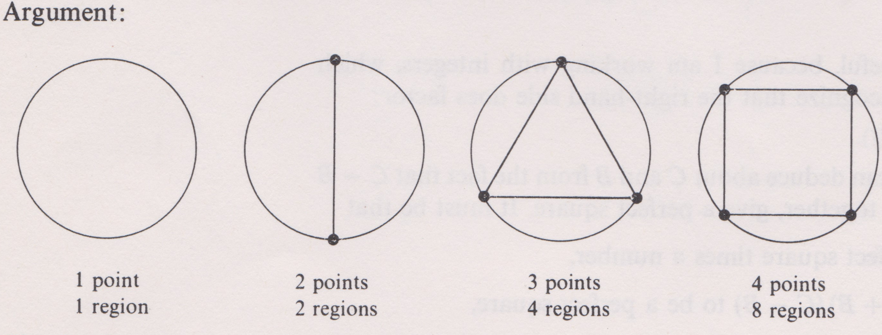
between men who are 4 feet 0.1 inch and men who are 4 feet 0.2 inch;

...

between men who are 6 feet 11.9 inches and men who are 7 feet.

5.11 The maximum number of regions that can be formed in a circle by drawing all possible chords between P points on the circumference is 2^{P-1}.

Argument:



Clearly, the pattern continues, doubling each time.

5.12 $\sqrt{2}$ is irrational.

Argument: Either $\sqrt{2}$ is rational or it is irrational. Suppose it is rational, so that $\sqrt{2} = p/q$, where p and q are integers; then $2q^2 = p^2$. The left-hand side has an odd number of prime factors, but the right-hand side has an even number, which is not possible, so $\sqrt{2}$ must be irrational.

5.13 If $A + B\sqrt{2} = C + D\sqrt{2}$ where A, B, C, D are integers, then $A = C$ and $B = D$.

Argument: $A + B\sqrt{2} = C + D\sqrt{2}$ implies $\sqrt{2}$ is a rational number unless $B = D$, in which case $A = C$.

The same argument generalizes to A, B, C and D rational numbers, and indeed real numbers, and $\sqrt{2}$ can also be replaced by $\sqrt{N}$ for any integer N.

5.14 The sum of any two even numbers is even.

Argument: $2N$ represents any even number, so $2N + 2N$ is the sum of any two even numbers, and that is $4N$, which is certainly even.

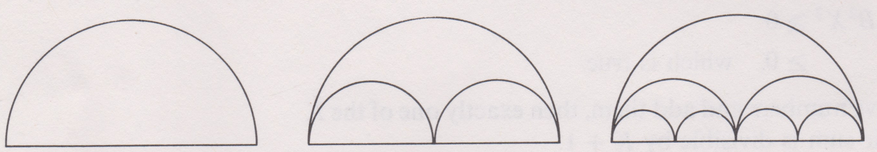
5.15 The sum of any two numbers is even.

Argument: N represents any number, so $N + N$ is the sum of any two numbers, and that is $2N$, which is certainly even.

5.16 One solution of the equation $\cos(\sin(x)) = x$, is $x = 1$.

Argument: My calculator gives $\cos(\sin(1)) = 1$.

5.17 $\pi = 2$.



Argument: In the figure, the diameter of the semi-circle is 2, and the perimeter of the semi-circle is π. Replace the semicircle by two semi-circles half the size of the first. Their total perimeter is still π. Replace them both by two semi-circles half their size, for which the total perimeter is again π. Keep doing this until there are so many semi-circles that they are virtually the same as the diameter. In the limit, the total perimeter must be the same as the diagonal, so $\pi = 2$.

5.18 Any solution in integers of $A^2 + B^2 = C^2$ must have the form of a multiple of the triple $X^2 - Y^2$, $2XY$, $X^2 + Y^2$ for some integers X and Y.

Argument: To begin, I know that

$$A^2 + B^2 = C^2,$$

and I want A, B and C all to be integers. I also want to be able to produce particular instances without having to guess, so what I really need to know is what relationships must exist between A, B and C apart from the given one. Since I have only one expression to work on, I try rearranging it:

$$A^2 = C^2 - B^2.$$

This immediately looks hopeful, because I am working with integers, which suggests factorizing, and I recognize that the right-hand side does factor:

$$A^2 = (C - B)(C + B).$$

Now I want to know what I can deduce about C and B from the fact that $C - B$ and $C + B$, when multiplied together, give a perfect square. It must be that

$C + B$ is itself a perfect square times a number,

and in order for $(C + B)(C - B)$ to be a perfect square,

$C - B$ is a perfect square times the same number.

More formally,

$$C + B = X^2 \times Z,$$
$$C - B = Y^2 \times Z,$$

and the same Z occurs in both because it is needed to make a perfect square for A^2.

Now, what do I want? I want to find relationships between A, B and C, and I know that $C + B$ and $C - B$ must have certain shapes expressed by X, Y and Z. I can solve for C and B in terms of X, Y and Z, then solve for A.

Some calculation gives

$$C = Z(X^2 + Y^2)/2$$
$$B = Z(X^2 - Y^2)/2,$$
$$A = XYZ.$$

What can be made of this? I notice that Z is a common factor for all three of A, B and C, so I can omit Z, and remember that I can always multiply each of A, B and C by any factor I like.

At the moment, C and B may not be integers, because they are both divided by 2. Why not get rid of that by using the Z! Put $Z = 2$ to get:

$$C = X^2 + Y^2,$$
$$B = X^2 - Y^2,$$
$$A = 2XY.$$

Having checked over my reasoning to look for slips, I now look back to see what I have learned ...

Have I shown that *every* integer solution of

$$A^2 + B^2 = C^2$$

must have the form of a multiple of the triple

$$X^2 - Y^2, \quad 2XY, \quad X^2 + Y^2?$$

Summary

Specializing and generalizing are available whenever you feel stuck. Being clear about what you KNOW and what you WANT can clarify the way ahead in an investigation, and can explain the giant leaps in a text. The growth of mathematical understanding comes from seeing connections and modifying intuitions as a result of specializing, generalizing and convincing. It is supported by a carefully cultivated atmosphere of conjecturing, and most importantly of all, it is the source of enormous pleasure.

INTERLUDE E ON EXAMPLES

Mathematics texts consist of definitions, techniques, results with proofs, and examples. What is the role of examples, and what are we really expected to do with them? Initially, this might seem a surprising question: since texts are littered with them, mathematicians must consider them important, and so the task of a student must be to learn what is in the text—or is it quite like that?

The words "learning" and "studying" are highly ambiguous. In one sense, "learn" and "study" can be interpreted to mean reading and re-reading until somehow the words and their meanings are inside me. They can also be interpreted to mean reconstructing the ideas in my own terms inside me, using the text for guidance. These interpretations can lead to very different activities.

One of the features of mathematical texts which makes them difficult to study is that examples are used for several purposes, but it is not always clear to the novice what those roles are. Hence there is a tendency either to treat them as distractions and pay them little attention, or to try to "learn" them like learning multiplication tables, and to treat all of the text as one large body of "knowledge" to be mastered.

Definitions, techniques and results tend to be treated identically, as "things to be learned", whereas the authors hope that the connections *between* examples, techniques and results are what readers will concentrate on, for it is the connections which contribute to understanding.

Techniques are things you can actually do. They are procedures which are used to answer standard types of questions, such as the bisection process for solving equations. The steps can be learned by memorizing, by practice, and through understanding what they do and how they do it. Techniques are the easiest things to test, and when they have been mastered they give a sense of confidence. But it is one thing to have a tool, and quite another to know when and how to use it. Examples are essential for this.

Results and their proofs (also known as theorems and propositions) are not simply the "facts" of mathematics. Some of them are merely technical results that are handy for use in proving other results, whereas the important ones are carefully-honed generalizations that mathematicians have worked over and over. They usually begin life as conjectures, vaguely stated, based on patterns and a sense of how things fit together. After some, and perhaps many modifications, they finally emerge in the form found in textbooks, with supporting arguments. If mathematicians have struggled to express and refine their perceptions, it is unreasonable to expect to be able to appreciate the result without at least some effort. This is, of course, where examples come in again.

Definitions are the building blocks of mathematics—presumably, therefore, they ought to come first, the way they tend to in books. For example, you cannot go far in mathematics without a clear idea of what a real number is, and what a function is. However, it took decades to move from a sense of what we now call a function, to a formal definition of it, and to some idea of its ramifications. The same is true of the real numbers, and of most other ideas. In fact, definitions most commonly arise in the middle of *doing mathematics*, as an attempt to reach a succinct formulation of some conjecture or some proof.

Examples play a key role in coming to grips with techniques, results, proofs and definitions. Examples (and exercises) are used to illustrate the steps of a technique—they act as specializations of the general process. To learn the technique effectively is to reconstruct my own version, by generalizing from the examples, guided by the general exposition. To know when a technique is applicable involves recognizing the sorts of questions which it answers. Again, examples provide the specializations which I must generalize for myself. If the examples all come from a narrow context, for example in the case of bisection if all the examples are polynomial equations, then I shall associate the technique with a narrow domain of application. If I appreciate the principle behind the technique, then the technique is more likely to surface when I need it in new contexts. In the case of bisection, equations involving trigonometric functions, logarithms and exponentials (to name only a few) are all amenable to the same method. If bisection as a process is appreciated, then whenever an equation has to be solved, there is a chance that bisection will be thought of as a possibility. It is a little like recognizing faces. It is often hard to recognize people I have met only briefly, when I meet them again in a new context. So, too, it is harder to "think" of using a technique in a new context if I have only a passing acquaintance with how it works and why.

In order for some piece of text to be an example, it must be an example *of* something. The author has in mind some general principle or theory, such as rules for manipulating inequalities based on a real feeling for numbers, their magnitudes, and a geometrical sense of the number line. Some examples are given, which for the author is an act of specializing. The first time I read them they are not yet examples *of* anything, but rather experiences of which some sense must be made, requiring acts of generalization. Of course, generalization by its very nature is uncertain—I may stress features that the author was ignoring, and vice versa. What seem like generalizations may turn out to be variations of some but not all the examples, and generalizations may turn out to be false statements, or not the ones the author has in mind. As a reader, I am constantly looking for

guidance as to which features are to be stressed, and which ignored, in order to come to what the author has in mind.

Sometimes, examples are used to give an introductory flavour of a topic, to illustrate the kinds of questions involved and to make the topic seem interesting. These sorts of examples should be carefully distinguished from other kinds, because they are not intended to be learned in the sense of memorized or mastered. They are only intended to set the scene, focus attention, and provide a flavour of what is to come. They are offered as part of an initial "see it go by". When the exposition begins in earnest, examples tend to be much simpler, cut to the bone so that extraneous detail does not get in the way of the reader's generalizing.

Examples play another role which is similar to, but not always the same as, the ones described so far. Whenever a mathematician is confronted by a general statement in a familiar area, there is an immediate response to specialize, to try it out on examples which have proved to be useful for this purpose. Sometimes, there is no single example, but a collection of examples. These familiar friends are the mathematician's touchstone, and have to be mastered inside and out so that they are available for confident manipulative use. When trying to recall a result, one way is to recall the examples, which then resonate with past experience and permit the result to be reconstructed.

Advice

Each time you meet an example in a text, ask yourself what it is exemplifying.

> Is it introductory?—in which case, read it for flavour.
>
> Is it demonstrating how a technique is carried out?—in which case, follow it through, paying attention to how it exemplifies the general technique, how it works, and what it does.
>
> Is it illustrating the meaning of a result?—in which case, follow it through with the general result in mind.
>
> Is it typical of the examples of a definition?—in which case, get to know it thoroughly.

Further Reading

1. Imre Lakatos. *Proof and Refutations*, Cambridge University Press (1976). A famous mathematical theorem is gradually clarified by means of a conversation in which arguments and objections are put forward by various characters linked to the historical development of the ideas.
2. George Polya, *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, Wiley, Combined Edition (1981).
   An expensive reprint, but by far the most important and useful book on mathematical problem solving. It investigates specializing and generalizing in depth with hundreds of examples.
3. John Mason, Leone Burton and Kaye Stacey. *Thinking Mathematically*, Addison Wesley (1982).
   Starting with specializing and generalizing, a framework is developed for improving mathematical thinking by learning from experience—complete with a psychological theory of how to learn from experience, and over a hundred problems to think about.
4. Philip J. Davis and Reuben Hersh. *The Mathematical Experience*, Harvester (1981).
   An excellent discussion that all students of mathematics should read.

RESOLUTIONS

Section 1 Specializing

1.1 The squares that I have available for numbers in the 30s are 1, 4, 9, 16, 25 and possibly 36. Can I use the same square twice? It doesn't say. Try 31, 32,... in sequence. I intend to subtract off a square, and see what can be done with the left-overs.

$31 = 25 + 6 = 25 + 4 + 1 + 1,$ four squares if repeats are permitted.

$31 = 16 + 15 = 16 + 9 + 6$
$= 16 + 4 + 11,$ neither of which look hopeful.

$31 = 9 + 9 + 9 + 4,$ a second way to do it.

$32 = 16 + 16,$ but I could find no other way.

$33 = 16 + 16 + 1$ using 32 that one was easy!
$= 25 + 4 + 4,$ now I can use these for 34.

$34 = 16 + 16 + 1 + 1$ (I am omitting all the intermediate calculations.)
$= 25 + 4 + 4 + 1$
$= 25 + 9$
$= 16 + 9 + 9$

$35 = 25 + 9 + 1$
$= 16 + 9 + 9 + 1$

$36 = 36$
$= 25 + 9 + 1 + 1$
$= 16 + 16 + 4$
$= 9 + 9 + 9 + 9$

$37 = 36 + 1$
$= 16 + 16 + 4 + 1$

and so it goes on. I have done more than asked, having found different ways of representing some of the numbers. I noticed that it was easy to get caught in a rut and not to notice other ways of getting a number as the sum of four or fewer squares. My specializing has suggested that it might be true that *any* number is the sum of four or fewer squares, but it is certainly false that three squares would suffice, since 31 requires four squares.

1.2 Try finding the possible error of one application. Is it one half of the starting value? What *is* the starting value?—what does it mean?

Bisecting the interval $[1, 2]$ gives $\frac{3}{2}$. The maximum error is $\frac{1}{2}$, because any number in the interval is at most $\frac{1}{2}$ away from the mid-point, $\frac{3}{2}$. Is the "starting value" the length of the interval? Try the interval $[1, 3]$, which is bisected at 2. The maximum error is 1, which is half the interval length. Yes, I see now that the mid-point of any interval admits an error of at most half the interval length. Each time I do a bisection, the error goes down by half. After four applications, the error will be $\frac{1}{16}$ of the original interval length. I deduce that "its" refers to the original interval length.

1.3 $\sqrt{1} = 1$
$\sqrt{11} = 3.3166248$ from my calculator
$\sqrt{121} = 11$
$\sqrt{1221} = 34.9428$
$\sqrt{12321} = 111$

This seems to suggest a pattern. I conjecture that 1 234 321 will have 1111 as its square-root, and that the square-root I was asked for is 111111111, because the number of 1s corresponds to the largest digit in the middle of the number. I find myself wondering what the square of ten 1s, 1111111111, might look like....

1.4 The only thing to do is to specialize.

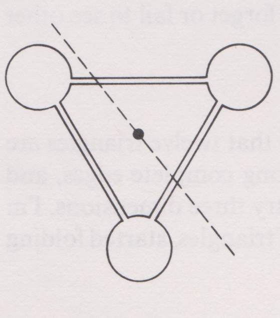
$1 + 2 + 4 = 7$ prime
$1 + 2 + 4 + 8 = 15 = 3 \times 5$ composite
$1 + 2 + 4 + 8 + 16 = 31$ prime
$1 + 2 + 4 + 8 + 16 + 32 = 63 = 3 \times 3 \times 7$ composite
$1 + 2 + 4 + 8 + 16 + 32 + 64 = 127$ prime
$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 = 255$
$= 3 \times 5 \times 17$ composite
$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 511$
$= 7 \times 73$ composite—oops!

Perhaps Tartaglia did not specialize far enough! How did I know to keep going? I have enough experience with primes to know that there are very few simple statements about primes which are likely to be true. The conjecture needs modifying, but it is not at all clear how to modify it.

N	$6N - 1$	$6N + 1$
1	5 prime	7 prime
2	11 prime	13 prime
3	17 prime	19 prime
4	23 prime	$25 = 5 \times 5$
5	29 prime	31 prime
6	$35 = 5 \times 7$	37 prime
7	41 prime	43 prime
8	47 prime	$49 = 7 \times 7$
9	53 prime	$55 = 5 \times 11$
10	59 prime	61 prime
11	$65 = 5 \times 13$	67 prime
12	71 prime	73 prime
13	$77 = 7 \times 11$	79 prime
14	83 prime	$85 = 5 \times 17$
15	89 prime	$91 = 7 \times 13$
16	$95 = 5 \times 19$	97 prime
17	101 prime	103 prime
18	107 prime	109 prime
19	113 prime	$115 = 5 \times 23$
20	$119 = 7 \times 17$	$121 = 11 \times 11$

Oops—neither of those last two are prime. Two things to learn from these examples: *never* believe your conjecture until you have found a convincing justification or at least feeling for *why* it might be true, because simply accumulating facts is not enough, and keep clear what you KNOW and what you WANT. I several times wrote down that a number was prime, 119 in particular, because I got into the habit of looking at a number and wishing it to be prime. I had to double check each time to be sure.

1.5 I tried squares and rectangles first. Then I tried a triangle and realized that I couldn't do the computations! Circles are clearly no help. I began to draw figures with long spindly bits joining massive bits. My favourite is shown here, but there are many other examples which show that the centre of gravity has nothing to do with lines bisecting area, even though it is a reasonable first conjecture from symmetric figures.



1.6 The true formula is

$$F = 32 + \tfrac{9}{5}C,$$

whereas the formula being offered is

$$F = 30 + 2C.$$

I experimented with various values for C, and found that for values between 5 and 15 degrees Centigrade, the two formulae give values that are no more than 1 degree Fahrenheit apart. The temperature in the UK is not confined to that range, but it is not a bad attempt.

A little bit of algebra (which you were not asked to do!) shows that

$$|(30 + 2C) - (32 + 9C/5)| < 1,$$

means that

$$|C/5 - 2| < 1,$$

or

$$-1 < C/5 - 2 < 1,$$

or

$$1 < C/5 < 3,$$

or

$$5 < C < 15.$$

In other words, *only* for values between 5 and 15 degrees Centigrade will the announced formula give values within 1 degree Fahrenheit.

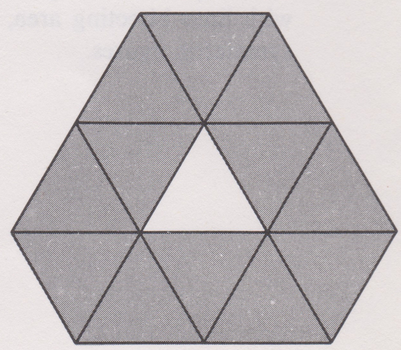
1.7 Specialize. If 2 divides the product of two numbers, say 14, then 2 certainly divides one of them, in this case because $14 = 2 \times 7 = 1 \times 14$. The same is true of 3 dividing a number, and more generally of any prime. The reason is that primes are indivisible, they cannot be broken up, so if a prime is to divide a number, it must divide one of the factors.

If I try a number that is not prime, like 4, then certainly 4 divides 2×2, but 4 does not divide either factor. The statement must be modified to refer to prime numbers dividing a product forcing the prime to divide a factor.

So far I have confined my attention to positive integers. What happens if rational or real numbers are admitted? It all depends on how I carry over the idea of "divides". With whole numbers it means divides exactly without remainder. If rational numbers are admitted, then 2 divides into 3 to give $\frac{3}{2}$, does it not? It seems best to leave the original statement referring to integers— though perhaps with some work the idea could be extended. You might like to consider what happens if you confine your attention to the set of numbers that have a remainder 1 when divided by 3. For example 4, 10 and 25 are in the set, and none is a product of members of the set other than itself with one, so 4, 10 and 25 are all "prime". However, $4 \times 25 = 10 \times 10$, so we find 4 divides 100 but 4 does not divide 10 or 10. You might like to investigate the "primes" in this system, always confining yourself to numbers having remainder 1 when divided by 3.

1.8 Trying various values of x, I first tried positive values, and found that the statement seemed correct. Negative values of x were fine until I tried $x = -4$. It is easy to be confined by what you want to be true, and to forget or fail to see other possibilities.

1.9 I specialized by drawing diagrams, and concluded that twelve triangles are needed, because triangles have to be glued together along complete edges, and there must be a proper hole. Then it occurred to me to try three dimensions. I'm not really sure what an annulus is, but I cut out a strip of triangles, started folding them, and discovered ... (you do it!).



Section 2 Generalizing

Dinner

I want some way to compare and combine the rates of eating. The lion eats $\frac{1}{2}$ of a sheep per hour, the wolf $\frac{1}{3}$ and the dog $\frac{1}{5}$. Thus in T hours, they have eaten

$T(\frac{1}{2} + \frac{1}{3} + \frac{1}{5})$ of a sheep.

I want T when one sheep has been eaten, i.e. when

$$T(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5}) = 1$$

or $$1/T = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5}.$$

I have carefully avoided doing arithmetical computations, because now I can generalize—replace 2 by L for the lion's time of eating one sheep, and so on.

Generalized Ratios

Instead of just adding numerators and denominators, it occurred to me to multiply by positive factors, to obtain things like

$$\frac{2A + 3C}{2B + 3D}, \quad \text{and more generally} \quad \frac{xA + yC}{xB + yD},$$

where x and y are positive.

It also occurred to me to use more ratios like E/F also equal to A/B, with positive factors:

$$\frac{xA + yC + zE}{xB + yD + zF}.$$

I confine myself to positive factors, because otherwise I might get a zero in the denominator.

It also occurred to me to let go of the equality restriction, and look at inequalities. For example,

$$\text{if } \frac{A}{B} < \frac{C}{D}, \text{ then where do } \frac{A + C}{B + D} \text{ and } \frac{2A + 3C}{2B + 3D} \text{ fit in?}$$

Exercise 4.5 investigates this further, but you might like to know that it lies at the heart of a good deal of Greek mathematics.

2.1 The first thing I did was to check the given statements. I seem to be faced with examples of numbers whose sum and product is the same, and one of the numbers is an integer. Having written down two more examples for myself, I generalized. Let A and B be two numbers of the sort I seek.

I know that $A + B = A \times B$.

I also know that A is to be an integer, so I want to find the corresponding B. Solving for B gives me

$$B = \frac{A}{A - 1}.$$

This formula specializes to the cases given, and enables me to generate other examples, not confined to integers. Similar work yields the second pattern, where the sum is the same as the product of one number and the square of the other. I can also see how to generate plenty of other similar examples.

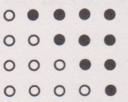
2.2 My first thought was to generalize to quadrilaterals—the sum of any three lengths must be greater than the fourth, otherwise they would not stretch far enough to complete the figure. This extends to larger polygons. It is not immediately clear to me whether this condition is enough to guarantee that a polygon can be formed, but it feels about right. Leave it as a conjecture.

My second thought was to move into three dimensions, and seek conditions on lengths of edges to be able to form a tetrahedron—that seemed hard, so I looked at what the triangle statement was saying, and switched to areas of triangles making up a tetrahedron. Presumably the sum of any three areas must be greater than the fourth in order to make a tetrahedron.

My third thought was to reverse the idea that two sides *must* be larger than the third, and I began to think about what sets of edge-lengths would fail to form any triangle at all. Eventually it emerged as the following.

> What would a set of k integers look like, which had the property that no three (distinct) members could be the edges of a triangle? How small could the largest element of that set be?

2.3 The 37th array will consist of 37 rows with $1, 2, 3, \ldots, 37$ dots in them, respectively. I actually want the sum of the first 37 positive integers. Before using standard techniques, let me investigate the suggestion. If I take two copies of one of the arrays, I can fit them together to form a rectangle which is R by $R + 1$, where R is the number of rows in the original array. (This requires some specializing!)



Thus two copies of the Rth array contain $R(R + 1)$ dots, from which I can get the standard formula for the number of dots in the Rth array, which is the same as the sum of the first R positive integers.

2.4 The pattern seems clear enough—until I try to write something down! If there are k 6s in front of a single 7, then I suspect that the square will consist of $(k + 1)$ 4s followed by k 8s and a final 9.

I checked the three examples and one more before my calculator switched into exponential notation, and so stopped me checking the pattern.

It is possible to justify the conjecture using knowledge of how to sum a geometric progression, or by carefully writing out rows of the long multiplication. You were not asked to do either! Here is the series approach.

$$67 = 6 \times (10 + 1) + 1$$
$$667 = 6 \times (100 + 10 + 1) + 1$$
$$6667 = 6 \times (1000 + 100 + 10 + 1) + 1$$

The number with k 6s to the left of a single seven, is one more than 6 times the sum of the first $k + 1$ powers of 10, starting at $10^0 = 1$. I can compute this sum, using the formula

$$1 + r + r^2 + \ldots + r^t = \frac{r^{t+1} - 1}{r - 1}$$

for the sum of a geometric progression, as

$$\frac{10^{k+2} - 1}{10 - 1} = \frac{10^{k+2} - 1}{9}.$$

I want to look at the square of

$$6 \times \frac{10^{k+2} - 1}{9} + 1 = 2 \times \frac{10^{k+2} - 1}{3} + 1,$$

which I shall call S for short. Then I want to find the pattern of 4s, 8s and 9 in S^2:

$$\left(2 \times \frac{10^{k+2} - 1}{3} + 1\right)^2$$
$$= 4 \times \frac{10^{2k+4} - 2 \times 10^{k+2} + 1}{9} + 4 \times \frac{10^{k+2} - 1}{3} + 1.$$

Now I look at what I want it to be, in order to see where to go from here. I want to end up with similar sums of geometric progressions for the repeated 4s, and the repeated 8s, so I want to get both denominators in the form 9, which is $10 - 1$. Furthermore, I am looking for terms of the form $10^{t+1} - 1$ coming from sums of powers of 10, so after some time looking at the mess of stuff, I find

$$4 \times \frac{10^{2k+4} - 2 \times 10^{k+2} + 1}{9} + 4 \times \frac{10^{k+2} - 1}{3} + 1$$
$$= 4 \times \frac{10^{2k+4} - 2 \times 10^{k+2} + 1}{9} + 12 \times \frac{10^{k+2} - 1}{9} + 1$$
$$= 4 \times \frac{10^{2k+4} - 1}{9} + 4 \times \frac{10^{k+2} - 1}{9} + 1,$$

which is another way of saying $(2k + 3)$ 4s added to $(k + 1)$ 4s added to 1. Adding them together gives $(k + 1)$ 4s followed by k 8s followed by 9.

2.5 This is very similar to the preceding exercise. In fact, the numbers being squared in 2.4 are one less than double the numbers being squared in 2.5.

2.6 First I check to see if the arithmetic is correct, and in the process of writing down the sums, begin to see patterns. I find it useful to write down the features which strike me—that the numbers are consecutive, that there is one more on the right-hand side than on the left of each one, and that there is one more term on each side than in the preceding line. This is enough to tell me how to write down the next few cases. Then I notice that the first term is a square... in fact, the square of the number of the line it is on. I am bold, and conjecture that the Nth line will begin with N^2. There will be N more terms on the left, and N terms on the right. (Specialize to check!) The result is a conjecture that

$$N^2 + (N^2 + 1) + (N^2 + 2) + \ldots + (N^2 + N)$$
$$= (N^2 + N + 1) + (N^2 + N + 2) + \ldots + (N^2 + N + N).$$

I notice that the next consecutive integer is $N^2 + 2N + 1$, which is a perfect square, ready to start the next line, so I feel fairly confident that I have the pattern right at least. I still do not know whether the two expressions are always equal!

You were asked only to generalize the pattern, but I notice that if I compare the two expressions, the last terms differ by N, as do the next to last, ... down to the second term of the first expression and the first term of the second expression. There are N terms in the second expression, so the second expression is N^2 more than the corresponding terms in the first, which has an extra N^2 to balance. The two expressions *are* always equal.

2.7 This one is similar in flavour to Exercise 2.6. First, I check the arithmetic, and at the same time look for patterns. I knew the first line, was a bit surprised by the second, and frankly dubious about the third, but they all checked. The features I noticed were that the left-hand sides have one more term than the right-hand sides, and that each line has one more term on each side than the preceding line. It is not so easy to see how to start each line. Let me call the starting number S. Then I suspect that the Rth row or line looks like:

$$S^2 + (S + 1)^2 + (S + 2)^2 + \ldots + (S + R)^2$$
$$= (S + R + 1)^2 + (S + R + 2)^2 + \ldots + (S + R + R)^2.$$

I checked that this does specialize back to the cases I was given for $R = 1$, 2 and 3—so now all I need are the starting numbers.

The first terms of the lines I am given are 3, 10, 21. These are rising by 7, then 11. A very bold guess might be to try a further rise of 15 on the grounds that 11 is 4 more than 7, but it seems pretty far-fetched. It also works!

$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$

How can I get hold of 3, 10, 21, 36? Try relating them to the number of the line that they start:

row	1	2	3	4
starts	3	10	21	36.

I notice that the row number divides the starting number:

$$3 = 1 \times 3, \quad 10 = 2 \times 5, \quad 21 = 3 \times 7, \quad 36 = 4 \times 9.$$

Staring at those factors for a while, and wondering how to deduce the second

factor, I noticed that

$$10 = 2 \times 5 = 2 \times (2 \times 2 + 1),$$
$$21 = 3 \times 7 = 3 \times (2 \times 3 + 1),$$
$$36 = 4 \times 9 = 4 \times (2 \times 4 + 1).$$

I am now ready to conjecture that the Rth row starts with $R(2R + 1)$. The Rth row should then read

$$S^2 + (S+1)^2 + (S+2)^2 + \ldots + (S+R)^2$$
$$= (S+R+1)^2 + (S+R+2)^2 + \ldots + (S+R+R)^2,$$

where $S = R(2R + 1)$. I checked to see that this does specialize back to the initial cases. I have got only a conjecture here, because I have not shown that the two expressions are always the same.

2.8 Having checked the arithmetic, I tried the same procedure on another pair of ratios and found the same result. With my experience with *Ratios* in mind, I generalized boldly!

Suppose $\dfrac{A}{B} = \dfrac{C}{D}$.

I conjecture that

$$\frac{pA + qB}{rA + sB} = \frac{pC + qD}{rC + sD},$$

where p, q, r and s can be positive, negative or zero as long as the denominators are not zero.

I know that $AD = BC$. I want two complicated ratios to be equal, so I want

$$(pA + qB)(rC + sD) = (rA + sB)(pC + qD):$$

left-hand side $= prAC + psAD + qrBC + qsBD$;
right-hand side $= prAC + psBC + qrAD + qsBD$.

The only difference between them is AD in place of BC, and vice versa. But $AD = BC$ was given at the start. Thus the conjecture is valid—as long as the denominators do not become zero. Thus because

$$\frac{4}{6} = \frac{10}{15}, \quad \text{I now know that } \frac{4 \times 2 - 6 \times 1}{4 \times 3 + 6 \times 7} = \frac{10 \times 2 - 15 \times 1}{10 \times 3 + 15 \times 7},$$

which is not at all obvious at first sight!

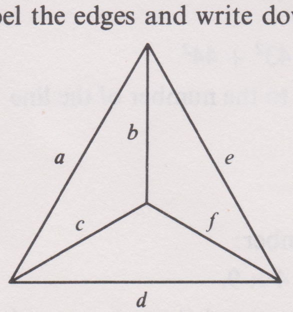
Section 3 Specializing and Generalizing Together

Recovering 8, 15, 17 from the general formula for Pythagorean triples

Since $2MN$ is even, it probably corresponds to 8, thus $MN = 4$. Try $M = 2$ and $N = 2$. No that was silly, because the first term of the triple would be zero. Try $M = 4$ and $N = 1$. The corresponding triple is 15, 8, 17 as requested.

Constant Perimeter Tetrahedra

First, I must be clear about the terms. A tetrahedron is a triangular-based pyramid. It has four triangular faces and six edges. Having drawn myself a picture, I could not see any way of using numbers effectively. The only thing I could see to do was to label the edges and write down what I knew.



Each edge belongs to two faces, so there is a fair amount of information floating

around. Let P represent the perimeter of each of the faces. Then

$$P = a + b + c,$$
$$P = a + d + e,$$
$$P = c + f + d,$$
$$P = b + f + e.$$

I WANT to see if there are any restrictions on the tetrahedron, and the particular perimeter is irrelevant, so I want to eliminate P. Before embarking, there must be some sort of pattern in the equations—for example, each letter occurs twice overall. Why not add them all together!

$$4P = 2(a + b + c + d + e + f)$$

I KNOW that $a + b + c = P$, so I can simplify and substitute in the last equation and get

$$2P = P + d + e + f,$$

or

$$P = d + e + f.$$

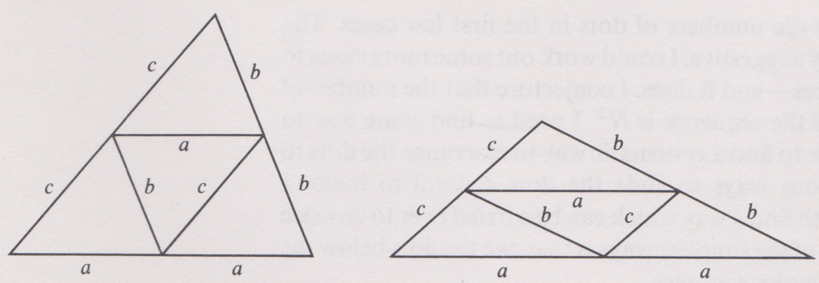
Now that is a new one—how does it compare with the original four? It overlaps quite nicely with three of them!

$$P = d + e + f,$$
$$P = a + d + e,$$
$$P = c + f + d,$$
$$P = b + f + e.$$

These immediately tell me that $a = f$, $c = e$ and $b = d$. Looking at the tetrahedron shows me that opposite edges of the tetrahedron must be equal... and more! The four triangles all have the same three edge-lengths, so all four faces are congruent.

Immediately, I must ask myself if such tetrahedra exist—surely I can make a tetrahedron from four copies of a triangle? Exercise 3.1 pursues this.

3.1 To make a tetrahedron, it seems awfully tedious to cut out four triangles and start glueing them together—I should be able to draw the triangles so as to be able to cut out a shape and fold the triangles into the tetrahedron. After some experimentation, I found that I was really starting with a triangle and dividing it into four congruent triangles.



The first few examples seemed to fold up without difficulty. Then I tried an obtuse-angled triangle, and it did not want to work. The flaps folded right over flat and still did not touch. I tried an approximately right-angled triangle, and the flaps just about touched. I am led to conjecture that the tetrahedron can be made only if the face triangles are acute-angled. I leave investigation of this conjecture until Exercise 4.2.

3.2 The four equations I am given involve the sums of three squares equal to a square, which can be expressed as

$$X^2 + Y^2 + Z^2 = W^2.$$

I want to be able to find integer solutions. The only thing I know that seems relevant is the Pythagorean triples. If Y, Z and W were one of those triples, then I could try to make either Y or Z the third member of another Pythagorean triple, and glue them all together.

I was unable to extend $3^2 + 4^2 = 5^2$, but I used it to extend

$$5^2 + 12^2 = 13^2 \quad \text{to} \quad 3^2 + 4^2 + 12^2 = 13^2.$$

It seems a cumbersome way to proceed.

I have been asked to do some algebra on A, B and C. By analogy with the Pythagorean triples, the two terms $(2AB)^2$ and $(2AC)^2$ are put in to convert the corresponding negative terms in the expanded square of the first expression, into the positive terms needed for the right-hand side. It is a generalization of the pattern for Pythagorean triples, and could even be extended, presumably, to more terms.

I specialized the formulas to produce the examples given, so the formulas do constitute a generalization! The last part claims that there are examples of three squares adding to a square which cannot be generated by the formula. Let us investigate.

I want to see why 3, 4, 12, 13 cannot be produced by the given formula. The formula has two even terms, so they must correspond to the 4 and the 12. Does it matter which one is which? Surely not, because B and C are interchangeable in the formula.

$$\text{Put } 4 = 2AB \quad \text{and} \quad 12 = 2AC,$$
$$\text{then } A = 2/B = 6/C,$$
$$\text{so } 2/B = 6/C, \quad \text{or} \quad B/2 = C/6 \quad \text{and} \quad C = 3B.$$

I also know that the first squared term must be 3, so substituting for C I obtain

$$3 = A^2 - B^2 - (3B)^2,$$
$$3 = A^2 - 10B^2.$$

I also know that 13 is the fourth term, so

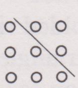
$$13 = A^2 + B^2 + (3B)^2,$$
$$13 = A^2 + 10B^2.$$

Putting the two equations in A and B together, I can eliminate B and find

$$2A^2 = 16, \quad \text{so } A^2 = 8.$$

However, A, B and C are supposed to be integers, so the given formula cannot generate the quadruple *3, 4, 12, 13*. I am left wondering if there *is* a general quadruple which will generate *all* possible integer quadruples, as an analogue with the Pythagorean triples.

3.3 Specialize by working out the numbers of dots in the first few cases. The numbers are 1, 4, 9, which is fairly suggestive. I could work out some more cases to see if the square pattern continues—and it does. I conjecture that the number of dots needed for the Nth term of the sequence is N^2. I need to find some way to justify this conjecture. One way is to find a systematic way to rearrange the dots to form a square. There are various ways to slide the dots around to make a square—though it is important to find a way which can be carried over to any size array, no matter how large. One of the simplest ways is to move the dots below the longest line up to the top and make a square.



The diagram shows a particular case, but illustrates what happens in every case.

3.4 Family Tree

I began by drawing myself some simple family trees and computing "parts". If my father is Cherokee and my mother Algonquin, then I am half Cherokee and half Algonquin. That much seems clear. Perhaps if my parents are part this and part that..., presumably my "parts" are found by averaging corresponding parts of my parents. After several specific calculations I realized that the only denominators I can possibly get are powers of 2. Unless Gilgamesh had not two parents, but three, I cannot see how the $\frac{2}{3}$ and $\frac{1}{3}$ can arise.

3.5 Crossless

The first thing to note is that the question is ambiguous. Are the points already given, and must the maximum be the maximum in the worst possible case for me, or am I permitted to place the points so as to maximize the number of non-crossing segments?

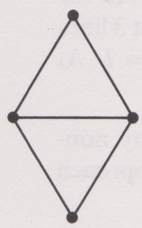
The latter seems to give more freedom, so I looked at it.

I want to arrange P points in the plane so that as many pairs as possible can be joined by straight-line segments, no two of which cross. Let L be the number of non-crossing lines. Specialize.

$P = 2 \quad L = 1$ no choice.

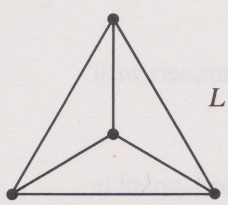
$P = 3 \quad L = 3$ no choice. Too early to conjecture.

$P = 4 \quad L = 5$



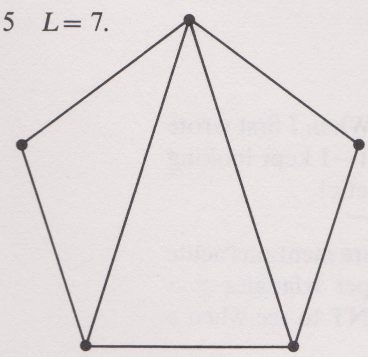
some choice, but expected $L = 6$.

AHA!

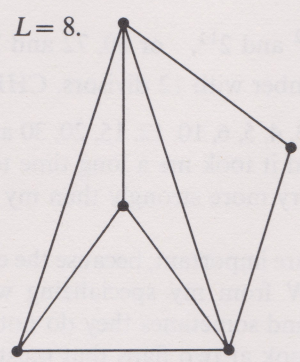


$L = 6$ by exploiting the choice.

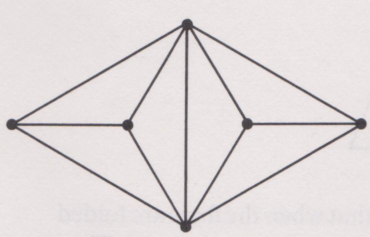
$P = 5 \quad L = 7.$



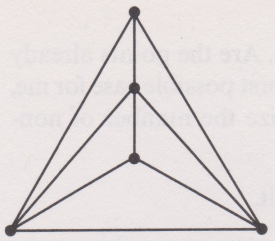
No! $L = 8.$



$P = 6 \quad L = 11$, glue two copies of $P = 4$ along an edge.



Looking at 1, 3, 6, 8, 11, I see that it alternates differences of 2 and 3. Why is that? $P = 4$ was the first real choice, and all possible lines are present. Try $P = 5$ again.



Aha! Put the fifth point *inside* one of the old triangles: $P = 5, L = 9$. Now I see a general pattern and a general strategy to support it.

Put each new point inside an old triangle and draw three new lines. For $P = 6$, $L = 12$ is possible this way. I predict that P points will produce ... the first two points are special, but after that there are three lines per point, so $1 + 3(P - 2) = L$ is my first (but wrong) conjecture. The initial 1 copes with the single line when $P = 2$. I must stipulate that my conjecture can only work for $P > 1$. Check it by specializing—it's wrong—so try again. For $P > 3$, I get 3 lines for each new point, and there are 3 lines for $P = 3$, so try $3 + 3(P - 3) = L$. At least this specializes correctly as long as $P > 2$!

Is it always going to work? My strategy can certainly yield that many non-crossing lines, but is that the very best possible? Might not some other approach yield more?

Section 4 Convincing Yourself and Others

4.1 Since $12 = 2 \times 2 \times 3$, according to the theory developed, the numbers with 12 divisors have the form

$$p^{2-1}q^{2-1}r^{3-1}, \quad p^{4-1}q^{3-1} \quad \text{or} \quad p^{12-1},$$

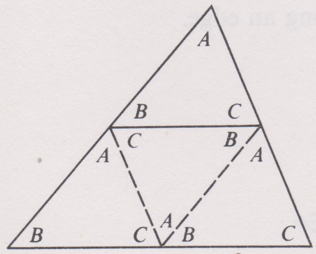
where p, q and r are distinct prime numbers. I WANT the smallest possible product, so I choose the smallest primes to go with the largest indices in each case. This leads me to

$$5 \times 3 \times 2^2,\ 2^3 \times 3^2 \text{ and } 2^{11}, \quad \text{or } 60, 72 \text{ and } 2048.$$

Thus 60 is the smallest number with 12 divisors. CHECK!

The divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60. When I first wrote them down, I forgot 20, and it took me a long time to find it—I kept looking because I believed my theory more strongly than my arithmetic!

4.2 I KNOW that angles are important, because the conjecture mentions acute and obtuse. I also KNOW from my specializing with paper triangles that sometimes the flaps meet, and sometimes they do not. I WANT to see when a tetrahedron is formed, so look at two flaps that must meet.



The angles B and C must together be large enough so that when the flaps are folded along the dashed edges, the edges marked 1 and 2 must actually meet in space. Thus B and C must add to more than the third angle at that point, which is A. I also KNOW that the sum of the angles of any triangle is 180°, and since two add to more than the third, the third must be less than 90°. Put algebraically,

$$\begin{aligned} &B + C > A \quad \text{for the flaps to meet properly,}\\ &A + B + C = 180, \quad \text{angle sum of any triangle,}\\ &A + A < 180, \quad \text{so } A < 90. \end{aligned}$$

The same thing happens between the other pairs of flaps, so all angles must be less than 90.° In other words, the triangle must be acute.

4.3 I KNOW that $\sqrt{2}$ is irrational.
I KNOW that $\sqrt{2}^{\sqrt{2}}$ is an irrational raised to an irrational power. Is that all there is to it? I WANT an irrational raised to an irrational to be rational. What about $\sqrt{2}^{\sqrt{2}}$? I don't know whether it is rational or irrational, though I suspect it is irrational. I have been told to look at the sequence, so let me continue my rather elementary specializing.

I KNOW that

$$\begin{aligned}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} &= \sqrt{2}^{\sqrt{2}\times\sqrt{2}}\\ &= \sqrt{2}^{2}\\ &= 2.\end{aligned}$$

This is certainly rational, but a bit of a surprise. I WANT an irrational raised to an irrational to be rational. Either $\sqrt{2}^{\sqrt{2}}$ is rational, in which case I have what I WANT, or it is irrational, in which case itself raised to $\sqrt{2}$, (which is irrational) is rational, which is what I WANT. In either case I get what I WANT, so the claim has been justified. Notice that I don't actually know *which* example is the one I want, but I do know that it must be one of them!

4.4 I KNOW that a rational number is a terminating decimal. I WANT a terminating decimal between two given rationals. Specialize:

try 0.1234 and 0.12567.

I can choose anything in between, say 0.12348, indeed anything finite on the end of the smaller number (generalizing). Can anything go wrong? Specialize again, trying to force my idea to go wrong:

try 0.12345 and 0.12346.

Not much room there—but I can still stick anything finite on the back of the smaller number. Hmm.

Try 0.1234 and 0.12340000001.

Reading the decimal places from left to right, eventually one of the digits of the larger number must be different from and bigger than the corresponding digit of the smaller number, and I can squeeze in my number with a digit in between. That's fine, but what about the repeating decimals—I had forgotten about them. It is still true that at some point the digits of the two numbers must disagree for the first time, and that is where I can prise them apart with a terminating decimal, which is rational. Later, someone showed me

1.49999999… and 1.5,

and asked me about inserting a number in between. My reply was that there *is* no number in between, that they are two representations of the same number!

Now, given two irrational numbers, I KNOW *only* that they have non-terminating, non-repeating decimal representations. I can still squeeze a number in between the way I did before—that argument made *no use* of the fact that the two given numbers were rational. I actually showed (generalizing) that between *any* two numbers, there is a rational number (indeed, a terminating decimal).

Given two irrational numbers, I WANT to squeeze an irrational in between—but what does an irrational look like? I KNOW only that it is non-terminating and non-repeating. I KNOW I can get a rational in between any two numbers, so perhaps I can change it into an irrational by giving it a tail. What sort of tail could I give it that was guaranteed not to repeat or terminate? I started writing down a sequence of digits:

101001000100001000001….

Each new 1 is followed by a longer sequence of 0s than has previously occurred, so there is no fear of repeating, and it goes on forever, so it doesn't terminate. All I have to do is stick that on the end of the rational which lies between the two given numbers, and I have what I WANTED.

I have actually shown that between any two numbers there is at least one rational number, and at least one irrational.

4.5 Specialize first!

$$\frac{1}{2} < \frac{3}{4},$$

so $\dfrac{2 \times 1 + 3 \times 3}{2 \times 2 + 3 \times 4} = \dfrac{11}{16}$ and $\dfrac{1}{2} < \dfrac{11}{16} < \dfrac{3}{4}$.

The best way to deal with the generality is to do general calculations. To show that one number is bigger than another, it is usually easier to subtract the smaller one from the larger.

I KNOW

$$\frac{2A + 3C}{2B + 3D} - \frac{A}{B} = \frac{2AB + 3BC - 2AB - 3AD}{B(2B + 3D)}$$

$$= \frac{3(BC - AD)}{B(2B + 3D)}. \quad \text{I WANT this positive.}$$

I KNOW that $\dfrac{A}{B} < \dfrac{C}{D}$, that is $\dfrac{C}{D} - \dfrac{A}{B} > 0$ or $\dfrac{BC - AD}{BD} > 0$.

So... I WANT a $(2B + 3D)$ on the bottom, and I have a D instead. Suppose all the numbers are positive—it will surely work then, because the denominators are positive forcing $BC - AD$ to be positive, so the difference I began with will be positive. It all hinges on $BC - AD$.

Now I want to see if I can defeat the inequality using negatives. Try

$$\frac{3}{-4} < \frac{1}{-2}, \quad \text{which produces} \quad \frac{2 \times 3 + 4 \times 1}{2 \times (-4) + 3 \times (-2)} = \frac{9}{-14},$$

and $\dfrac{3}{-4} < \dfrac{9}{-14} < \dfrac{11}{-2}$, which is fine.

I WANT a minus sign on top as well, say in C.

$$\frac{3}{-4} < \frac{-1}{2} \quad \text{produces} \quad \frac{2 \times 3 + 3 \times (-1)}{2 \times (-4) + 3 \times 2} = \frac{3}{-2},$$

and defeats the conjecture.

This shows that the original assertion has to be modified. I shall be content with assuming that A, B, C and D are all positive. There might easily be a refinement that permits negatives, but I shall leave it there.

Looking back over my work, I discovered that I entirely forgot about the other half of the inequality. When I worked it through, there were no more surprises, fortunately!

Trying to place it in a more general context, I replaced the 2 and the 3 in the original question, with X and Y, both assumed positive. I expected to have to place another condition on X and Y, but the algebra showed it to be unnecessary.

A journey consisting of two parts could result in an average speed of A/B for the first part, and an average speed of C/D for the second part. The average speed overall must lie between the average speeds for the two parts—a re-statement of the conjecture. Inserting X and Y just magnifies the distances and time in each part, but still the average overall lies between the averages for each part.

The parallelogram $(0,0)$, (XB, XA), (YD, YC), $(XB + YD, XA + YC)$ must have the slope of the diagonal from $(0,0)$, between the slopes of the edges from $(0,0)$, which is a re-statement of the conjecture.

Section 5 When is an Argument Valid?

One is the largest whole number

In the case of Steiner, it can with difficulty be shown that there is a largest area, and the problem is to show that this largest area is in fact a segment of a circle. In the largest number conjecture, the argument breaks down when it is claimed that you must eventually get close to one. This claim already assumes that the original conjecture is true, namely that there is a largest number.

5.1 While it is certainly true that log is an increasing function, it pays to be clear about the domain of a function before beginning work on it. In this case, $\sin(x)$ always lies in the interval $[-1, 1]$, and log can accept only positive values for its domain. Since the log of a number less than or equal to 1 is at most 0, the second log function cannot accept any of the images of $\log(\sin(x))$. Thus the composite function has no domain at all.

5.2 The previous example suggests checking the domain of the function, and again there are no real values of x which, when squared and added to 2, are acceptable to $\sin^{-1}$.

5.3 The argument offered is somewhat succinct. I WANT to know what x and y must be. I KNOW that they satisfy the equation. I WANT two squares to add to 0. Since squared numbers are never negative, they must both be 0 in order to add to 0. Therefore $x = 2$ and $y = -3$.

5.4 The argument offered is argument by specializing. In this case, *all* possible cases have been considered: since all arithmetic is being done modulo 3, any number is congruent to 0, 1 or 2 modulo 3. The argument is valid as it stands, but compare it with Exercise 5.11, which tries to argue the same way!

5.5 A great deal of care is needed when writing down equations from words. It is tempting to catch sight of "14 times ... students", but in fact the words omitted, "as many as", also have meaning. It may help to translate the words into more careful English first, paying particular attention to what it is that S and P stand for.

The number of students is 14 times the number of professors.

Now it is much easier to translate into symbols because all the English words have mathematical versions, whereas there is no mathematical version of "as many as".

5.6 This is an example of "argument by calculator". The number is an integer only to the accuracy of the calculator. It is not, in fact, an integer.

5.7 The argument offered needs expanding with KNOW and WANT, but it shows that the edges of the octagon are all equal in length. The claim is that the octagon is regular—what does regular mean? It means equal sides and equal angles. Further computation is required! Unfortunately, the angles turn out to be slightly different.

5.8 The proposed argument starts with what it WANTS and works away, modifying the WANTS, until it reaches what is KNOWN. A convincing argument may indeed involve modifying what is WANTED, but in the final presentation of the argument, it must be clear that, starting with what is KNOWN, it is possible to reach what is WANTED. In this case, it is possible to add words to the steps to make the argument convincing by labelling clearly with WANT and KNOW. An alternative presentation, which is easier to follow, is obtained by reversing the steps, and again labelling clearly what is KNOWN at each stage. Reversing the steps does not always work though. Try following through the revised argument with the above steps reversed, but A, B, X, Y negative.

5.9 Look first at the case when $K = 2$. The argument as given seems to do little more than restate what is WANTED. I WANT two consecutive numbers, so N and $N + 1$ seem reasonable choices. Now I WANT their sum, which is $2N + 1$. So far so good. Now I WANT to show that exactly one of these numbers is divisible by $K + 1$, which is 3. I detect three cases:

$$N \equiv 0 \bmod 3; \quad N \equiv 1 \bmod 3; \quad N \equiv 2 \bmod 3.$$

In each case, I compute $N + 1$ and $2N + 1$ to find that exactly one of N, $N + 1$, $2N + 1$ is divisible by 3. That much of the argument is reasonable.

The proposed argument goes on to claim that a similar argument works for other K. It seems a little early to generalize, since I have only one case to go by, so I specialize further.

Try $K = 3$. I choose 1, 2, 3, which seem like good candidates for three consecutive numbers, and their sum is 6. None of them is divisible by $K + 1$, which is 4. So much for that argument. I am led to wonder if it does work for any other value of K, but that is another question!

5.10 The result is certainly wrong—no, let me be careful—the result involves a curious meaning of "no difference between". This example demonstrates the necessity of having technical terms which are clear, and which correspond to intuition. There is an underlying assumption that if "no difference" is added to "no difference", then the result will still be "no difference", which seems implausible!

5.11 There is nothing wrong with the specializations offered, but one must always be careful about generalizations. They make good conjectures, but they need supporting arguments before they are convincing. In this case, further specialization (two more cases, not just one!) shows that the pattern is *not* powers of two; it just looks like it at first.

5.12 The argument here is perfectly sound, but it relies without comment on the observation that the *number* of prime factors of an integer is a meaningful idea—so that it is not possible to get two different values depending on how you set about finding the prime factors. If that seems too obvious, try looking for prime factors of numbers of the form $3N + 1$, where all factors must also be of the same form. In other words, the *only* numbers you know about have the form $3N + 1$. See the resolution of Exercise 1.7.

5.13 The argument offered, while succinct, can be expanded by insertion of what is KNOWN and WANTED, to make it clearer. The generalization gratuitously thrown in at the end seems overly ambitious! Having written out the argument in more detail, it does extend to A, B, C and D all rational numbers but no further, because a crucial step in the argument is that $\sqrt{2}$ is not rational. (In fact, the argument does extend to other sorts of numbers, such as $\sqrt{3}$, but is too complicated to pursue here.)

5.14 It is certainly true that $2N$ represents any even number. However, in order to show that the sum of *any* two even numbers is even, I have to find a way to represent *any* two even numbers. The argument proposed adds an arbitrary even number to itself. To correct the argument, I WANT a representation of two perfectly arbitrary even numbers. One of them can be $2N$ and the other could be $2M$. Now the argument works. Note that as a special case of the argument, M and N might have the same value, but the presence of the N and M means that they might also have different values.

5.15 This is another example of the same error as in Example 5.12, but this time the assertion is false. For example $1 + 2 = 3$, which is not even.

5.16 My calculator gives $\cos(\sin(1)) = 1$ only when it is in degree mode. Working in degrees, it is not clear what to make of $\sin(1)$, because $\sin(1)$ is a number and not a number of degrees. Does composition of cos and sin really make sense when the calculator is in degree mode?

5.17 The argument offered purports to prove that two numbers are equal. The bridge between the two numbers comes from a diagram. To get π, I am told to look at the length of the semi-circular arcs, and at each stage it is hard to dispute that the total remains constant at value π. To get 2, I am told to look at the length of the diameter of the first semi-circle, and to agree that the arcs appear to be getting closer and closer to the diameter.

My faith in the argument rests on my sense of "gets closer and closer". Look at the total area encompassed by the circular arcs at each stage. The first arc has area $\pi/2$. The second pair of arcs encompass a total area of

$$2 \times (1/2)^2\pi = \pi/2,$$

and so it continues. At every stage, the arcs still encompass the same area as the original, which seems to go against my intuitive feelings for what "getting closer and closer" actually means. Examples like this acted as a spur to mathematicians to clarify what we mean by two curves getting closer and closer to each other. The result is a batch of definitions and theorems forming a chapter of mathematics.

5.18 The argument begins by KNOWING that $A^2 + B^2 = C^2$, and deduces relationships which must necessarily hold. The only doubtful bit is when Z is made equal to 2. If X and Y are both odd or both even, then

$$C = Z(X^2 + Y^2)/2,$$
$$B = Z(X^2 - Y^2)/2,$$
$$A = ZXY,$$

are all integers already. Putting $Z = 1$ will give me a triple which might not arise by putting $Z = 2$ and choosing X and Y differently. For example, with $X = 7$ and $Y = 3$,

$$C = (X^2 + Y^2)/2 = 29,$$
$$B = (X^2 - Y^2)/2 = 20,$$
$$A = XY = 21.$$

However, it turns out that 20, 21, 29 *can* be produced by the triple

$$X^2 - Y^2, \quad 2XY, \quad X^2 + Y^2,$$

by putting $2XY = 20$ and choosing $X = 5$ and $Y = 2$.

In fact, by going back to an early stage of the argument, the hiatus can be repaired. If $A^2 + B^2 = C^2$, and all common divisors have been removed, one of A or B must be even, but not both. (If neither, C^2 is divisible by 2 but not 4; if both, then C is also even.) Let A be the even one.

Now proceed as before. Since B and C are odd, $C + B$ and $C - B$ are both even. Let $A = 2a$.

Then $A^2 = 4a^2 = (C + B)(C - B)$,

$$a^2 = \frac{C+B}{2} \times \frac{C-B}{2}.$$

Then $\dfrac{C+B}{2} = X^2Z$,

$$\frac{C-B}{2} = Y^2Z$$

and $a^2 = X^2Y^2Z^2$.

Since A, B and C have no common factors, Z must be 1.

Then $C = X^2 + Y^2$,

$$B = X^2 - Y^2,$$
$$a = XY,$$
$$A = 2XY,$$

as before. Now I am sure that *all* Pythagorean triples are generated by

$$X^2 - Y^2, \quad 2XY, \quad X^2 + Y^2.$$

You might like to investigate $X^2 + Y^2 + Z^2 = W^2$ as in Exercise 3.2.

For your amusement!

A

The statement in box B on this page is true

B

The statement in box A on this page is false

C

The statement in box C on this page is false

D

There are two misstakes in the statement in box D

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